DOCUMENT RESUME

ED 174 449 SE 028 513

TITLE Biomedical Mathematics, Unit VII: Exponential and Logarithmic Functions. Student Text. Revised Version, 1977.

INSTITUTION Biomedical Interdisciplinary Curriculum Project, Berkeley, Calif.

SPONS AGENCY National Science Foundation, Washington, D.C.

PUB DATE 77

NOTE 91p.; For related documents, see SE 027 978-999 and SE 028 510-516; Not available in hard copy due to copyright restrictions; Contains occasional light type

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.

DESCRIPTORS Career Education; Diseases; Drug Education; Environmental Education; Graphs; Health; *Health Education; Interdisciplinary Approach; Mathematical Applications; Mathematical Concepts; *Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; Radiation; Radiation Effects; *Science Education; *Secondary Education

IDENTIFIERS *Exponential Functions; *Logarithms

ABSTRACT

This collection of lessons, exercises, and experiments deals with exponential and logarithmic mathematical functions in the context of biomedical situations. Typical units in this collection provide discussion of the biomedical problem or setting, discussion of the mathematical concept, several example problems and solutions, and a set of problems for the student to solve. Seventeen concept sections are presented, each subdivided into several lesson units. (RE)

BIOMEDICAL MATHEMATICS

UNIT VII

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

STUDENT TEXT

REVISED VERSION, 1977

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

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SECTION 1: EXPONENTIAL FUNCTIONS

1-1 Exponential Functions of the Form $y = r^t$

In the Biomedical Mathematics course you have seen many examples of the use of functions. Both linear functions and quadratic functions were discussed in detail. In this unit you will work with the most important type of function that appears in biomedical settings, the exponential function. From an applied point of view, we will find that exponential functions will describe many different kinds of events: for example, the spread of diseases, the metabolism of a drug in the bloodstream and the decay of radioactive substances.

The growth of a bacteria culture provides a simple example of an exponential function. Suppose that we begin with a single bacteria cell, which at the end of one day splits into two cells. When another day has passed, both these cells split, and the process continues as long as sufficient food is available. A table can be prepared which shows the number of cells as a function of days passed.

t (days)	y (number of cells)
0	1
1	2
2	4
3	8
4	16
5	32
6	64

The numbers that appear in the y column are just the integral powers of 2. That is, $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$ and so on. In fact at the end of t days there are 2^t bacteria cells. The function displayed in the table can be described by the equation

$$y = 2^t$$

The reason for the term exponential function can be seen in this equation. One of the variables appears as an exponent.

1-2 Exponential Functions of the Form $y = Ar^t$

In the example just discussed the exponential function had a formula of the form $y = r^t$ with $r = 2$. A slightly more complicated formula is the following, that describes the growth of a bacteria culture with a starting population of 100 cells. The population triples in size each day.

$$y = 100\cdot(3^t)$$

This formula is of the form

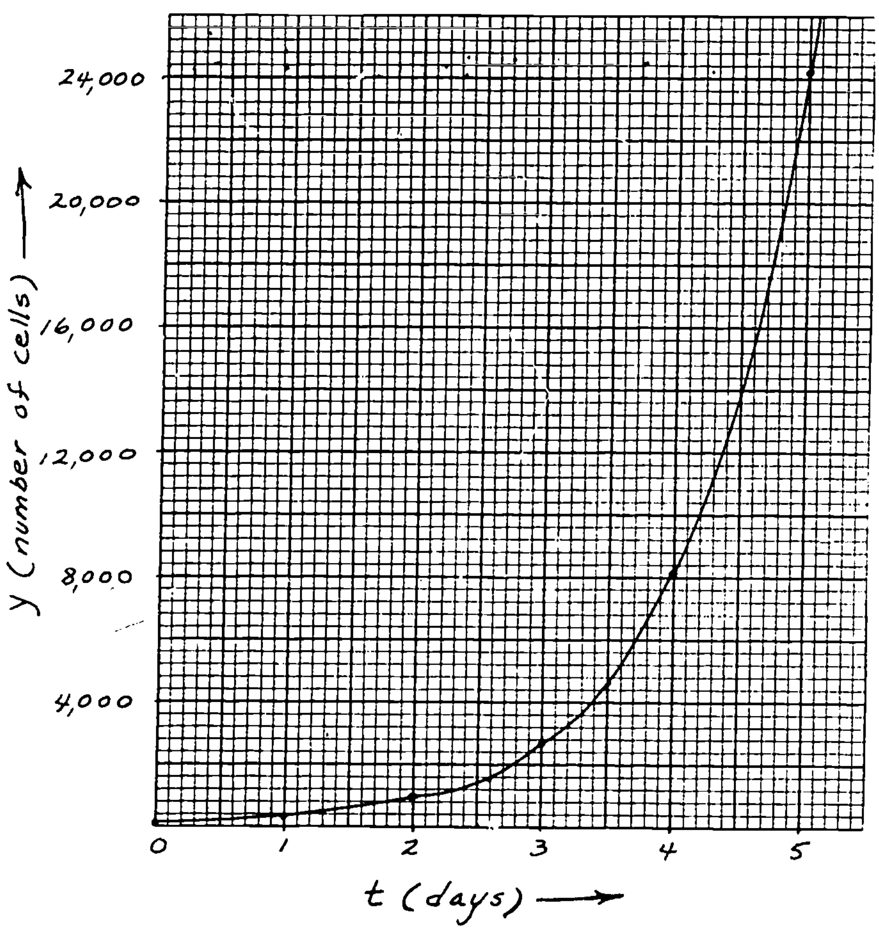
$$y = Ar^t$$

with A = 100 and r = 3. The number A is called the <u>scaling constant</u> and the number r is called the <u>base</u> or the <u>common ratio</u>.

A table of values for the function is shown below.

t (days)	y (number of cells)
0	100
1	300
2	900
3	2700
4	8100
5	24300

A graph displaying these data is shown below. Notice that the points have been joined by a smooth curve. Actually this misrepresents the situation because only integral y-values are possible. A fractional part of a bacteria cell makes no sense. This must be kept in mind while interpreting the graph.



EXAMPLE:

Determine the number of bacteria present at the end of 3.7 days.

SOLUTION:

The graph can be used to make a rough estimate. It appears that y will be about 6000.

In order to calculate the number we return to the formula $y = 100\ (3^t)$. Since $t = 3.7$ we must determine $100(3^{3.7})$. The value of $3^{3.7}$ can be found easily by using the logarithm table inside the back cover of this book. The calculations follow.

$$\begin{aligned} \log 3^{3.7} &= 3.7 \log 3 \\ &\approx 3.7\ (.477) \\ &\approx 1.765 \end{aligned}$$

From the log table we find that $10^{1.765} \approx 58$. Therefore

$$3^{3.7} \approx 58$$

Finally

$$\begin{aligned} y &\approx 100\ (3^{3.7}) \\ &\approx 100(58) \\ &\approx 5800 \text{ bacteria cells.} \end{aligned}$$

EXAMPLE:

For what value of t will the population be 1,000,000?

SOLUTION:

The problem is to find t such that

$$100\ (3^t) = 1{,}000{,}000$$

or, equivalently

$$3^t = 10{,}000$$

Taking logarithms of both sides we obtain

$$\begin{aligned} \log 3^t &= \log 10{,}000 \\ \log 3^t &= 4 \\ t \log 3 &= 4 \\ t &= \frac{4}{\log 3} \end{aligned}$$

Therefore

$$\begin{aligned} t &\approx \frac{4}{.477} \\ &\approx 8.4 \text{ days.} \end{aligned}$$

1-3 Growth Rates of Exponential Functions

Upon inspecting the graph of Section 1-2 it is evident that the function is not growing at a constant rate. As the value of t increases, the growth is more rapid (the slope is steeper). This is to be expected since a larger cell population means that more new cells will be produced. The rate of increase in the number of cells depends on how many cells are already present. This is an important property of all exponential functions. We can learn more by looking at a couple of specific examples. Below are tables of values for the two exponential functions $y = 100(3^t)$ and $y = 100(5^t)$. In each case a Δy column has been added to display the step-by-step growth in the value of y.

$y = 100(3^t)$

t	y	Δy
0	100	200
1	300	600
2	900	1800
3	2700	5400
4	8100	16200
5	24300	

$y = 100(5^t)$

t	y	Δy
0	100	400
1	500	2000
2	2500	10000
3	12500	50000
4	62500	250000
5	312500	

Look at the table on the left. Can you see a relationship between the y-column and the Δy-column? Each entry in the Δy-column is twice the entry just above it in the y-column (i.e., $200 = 2 \cdot 100$, $600 = 2 \cdot 300$, $1800 = 2 \cdot 900$, etc.). Hence we can write

$$\Delta y = 2y$$

This is simply a statement of proportionality. It says that the growth of y is directly proportional to y. The proportionality constant is 2.

What about the table on the right? Each entry in the Δy-column is four times the entry just above it in the y-column (i.e., $400 = 4 \cdot 100$, $2000 = 4 \cdot 500$, etc.). We can write

$$y = 4y$$

This is also a statement of proportionality; only the proportionality constant is different. In order to see how the proportionality constants arise we write each growth equation below its exponential function.

$$y = 100(3^t) \qquad y = 100(5^t)$$

$$\Delta y = 2y \qquad \Delta y = 4y$$

These examples suggest a relationship, namely that the proportionality constant is one less than the base of the exponential function. This turns out to be true for all exponential functions.

The growth equation for the exponential function

$$y = Ar^t$$

will be

$$\Delta y = (r-1)y.$$

For an exponential function the rate of growth of y is proportional to y. This important property will allow us to "guess" when to use exponential functions in describing data sets.

PROBLEM SET 1:

Problems 1 through 4 are concerned with the function of $y = 10^t$.

1. Construct a table displaying the values of the function for integral values of t from 0 to 6.

2. Add a Δy column to the table and fill in the values.

3. For this function $\Delta y = Ky$. K = ?

4. Use the log table at the back of the book to find y when t = 3.6.

The population of a certain bacteria culture is given by the equation

$$y = 1000(2^t)$$

where t is expressed in hours.

5. What is the base of the exponential function?

6. What is the scaling constant of the exponential function?

7. What is the initial population at time zero?

8. Construct a table displaying the values of the function for integral values of t from 0 to 5. Include a Δy column in the table.

9. What is the common ratio between successive entries in the y column?

10. The common ratio is the same as the <u>(?)</u> of the exponential function.

11. For this function $\Delta y = Ky$, K = ?

12. Draw a graph from the entries in the table and connect the points with a smooth curve.

13. Use the graph to estimate the bacteria population when t = 3.4 hours.

14. Use the log table at the back of the book to compute the bacteria population when t = 4.7 hours.

15. Use the log table to compute the time at which the population will reach 20,000.

Write a growth equation $\Delta y = Ky$ for each of the following exponential functions.

16. $y = 3^t$

17. $y = 5^t$

18. $y = (\frac{5}{2})^t$

19. $y = 862^t$

20. $y = 30(18)^t$

21. $y = 128(59)^t$

The table shown opposite shows the first few values of an exponential function. When y is 13, Δy is 26. When y is 39, Δy is 78 and so on.

t	y	Δy
0	13	
		26
1	39	
		78
2	117	
		234
3	351	

22. What will Δy be when y is 1053?

23. What will Δy be when y is 255,879?

If a bacteria culture is started with one cell of E. coli and adequate nutrients are available, the population after t hours will be given by

$$y = (11.5)^t$$

24. Use the log table at the back of the book to estimate the population at the end of 5 hours.

*25. If each E. coli cell occupies an area of 1 μm^2, compute the area in km^2 covered by the cell population at the end of 15 hours.

*26. How many hours must pass before the population covers an area equal to the earth's surface? You will need to use the following information.

Diameter d of the earth ≈ 12740 km.

Formula for area of a sphere $A = \Pi d^2$

use $\Pi \approx \frac{22}{7}$

27. Why is it that such a huge bacteria population never devleops in a real situation?

SECTION 2: GROWTH RATES OF EXPONENTIAL FUNCTIONS

2-1 A Decreasing Exponential Function

In the last section we described a fundamental property of any exponential function. The rate of growth of the function is proportional to the value of the function. In equation form this is written as

$$\Delta y = Ky$$

where K is a proportionality constant which depends on the particular function we are discussing.

We also noted that the proportionality constant K is one less than the base, r, of the function. Both of the examples of the last section were increasing exponential fucntions. Now we will look at an example of a decreasing exponential function and see if our observations still hold true.

Suppose that a population of 1000 organisms is being killed off by predators. At the end of t days the number, y, of individuals remaining is predicted to be

$$y = 1000\ (.7)^t$$

Here the scaling constant is 1000, the population when t = 0. The base is .7. The table on the following page shows the results predicted for the first few days. A Δy column has been included to show step-by-step growth.

Notice that now the values of Δy are negative because the values of y are decreasing with the passing of each day. What is the relationship between Δy and y for this table? In order to answer this question we note the following pattern.

t	y	Δy
0	1000	-300
1	700	-210
2	490	-147
3	343	

$$-300 = (-.3)1000$$

$$-210 = (-.3)700$$

$$-147 = (-.3)490$$

This indicates that

$$\Delta y = (-.3)y$$

and hence the proportionality constant is -.3.

Is the proportionality constant one less than the base as in the previous examples we have considered? Recalling that the base is .7, we find that

$$.7 - 1 = -.3$$

Therefore the relationship holds for this function as well. In fact, it holds for all functions of the form $y = Ar^t$.

2-2 Finding Equations for Exponential Functions

All the previous exploration of growth rates of exponential functions will be useful to us as we try to decide when an exponential function is the proper type to describe a particular set of data. Whenever the growth rate of a function seems to be proportional to the value of the function, we have good reason to think we are dealing with an exponential function. Moreover, the relationship between the base and proportionality constant which we have just discovered is sometimes helpful in finding the actual equation of the exponential function. The following examples will illustrate the process.

The growth of a savings account is an example of proportional growth. Banks pay interest. The amount they pay is proportional to the amount that is in the savings account. In fact the interest paid is a certain percentage of the amount of money in the account.

EXAMPLE:

Suppose that an account is started with a $1000 deposit and no other deposits or withdrawals occur. If the interest rate is 5%, write an equation giving the value of the account at the end of t years.

SOLUTION:

If the account contains y dollars then the interest at the end of the year will be (.05)y. Thus the account will grow by an amount Δy given by

$$\Delta y = (.05)y$$

Remembering that the proportionality constant is one less than the base of the exponential function, we deduce that the base must be 1.05. Therefore the exponential function must be of the form

$$y = A(1.05)^t$$

and we need only find the scaling constant A. To do so, we note that when $t = 0$ the account contained $1000. Substituting,

$$1000 = A\ (1.05)^0$$

$$= A \cdot 1$$

Therefore $A = 1000$ and the function is

$$y = 1000\ (1.05)^t$$

EXAMPLE:

The opposite table concerns an imaginary cell colony which is being attacked by a disease. Determine if an exponential function will fit this table and if so find it.

t(min)	y(number of living cells)
0	1,000,000
50	900,000
100	810,000
150	729,000
200	656,100

SOLUTION:

In all our previous tables the time units have been 0, 1, 2, 3, etc. We can put the above table in a similar form by letting one time unit equal 50 minutes. A Δy column has also been added.

t (50 min)	y (number of living cells)	Δy
0	1,000,000	
		-100,000
1	900,000	
		-90,000
2	810,000	
		-81,000
3	729,000	
		-72,900
4	656,000	

There are two ways to see that an exponential function will describe the data. One way is to observe that consecutive pairs of y-values are always in the ratio .9 ($\frac{900,000}{1,000,000} = .9$, $\frac{810,000}{900,000} = .9$, etc.). A common ratio of .9 indicates a base of .9, so the function must be of the form

$$y = A(.9)^t$$

By the method of the last example, the scaling constant A just turns out to be 1,000,000, the cell population at time zero. Therefore,

$$y = 1,000,000(.9)^t$$

This formula can also be derived by noting that each value in the Δy column is negative one tenth of the y value just above it. That is

$$\Delta y = (-.1)y.$$

This proportional growth indicates an exponential function. The base must be one plus the proportionality constant.

$$r = 1 + (-.1)$$
$$= .9$$

Thus

$$y = A\,(.9)^t$$

and as above A is found to be 1,000,000.

The two examples we have just considered are simpler than many in one respect. We do not have to deal with the problem of error. The bacteria problem was fanciful because the equation fits the data exactly. If the bacteria populations had been obtained from measurements, we would not have been so fortunate as to obtain an exact fit. In future lessons you will see how to fit data sets in which error is involved.

PROBLEM SET 2:

Problems 1 through 5 concern the exponential function

$$y = 100(.8)^t$$

1. Prepare a table displaying the values of y as t ranges from 0 to 4.

2. Add and fill in a Δy column. Pay attention to the signs of the Δy entries!

3. Inspect the y and Δy columns. What is proportionality constant K in the growth equation

$$\Delta y = Ky \text{ ?}$$

4. What is the base r of the exponential function?

5. What is the relationship between the proportionality constant K and the base r?

Write a growth equation $\Delta y = Ky$ for each of the following functions. It is not necessary to prepare a table of values.

6. $y = 20^t$

7. $y = (1.1)^t$

8. $y = (1.01)^t$

9. $y = (.4)^t$

10. $y = 13(.6)^t$

11. $y = 19(\frac{1}{2})^t$

12. $y = 100(.01)^t$

13. $y = \frac{82}{3}\,(\frac{1}{11})^t$

14. A savings account pays 10% per year. The initial amount is $500.

a. Write a growth equation $\Delta y = Ky$ for the account.

b. Write an exponential function giving the amount in the account as a function of years t.

15. A particular colony of bacteria is known to grow at the rate of 8% per hour during its "takeoff" phase.

a. Write a growth equation for the bacteria population.

b. Write an exponential function giving the population y in terms of hours t. Assume the initial population to be 700.

16. A bacteria culture at a temperature of 30 °C is placed in a refrigerator at 0 °C. The Celcius temperature of the culture then decreases by 10% per minute.

a. Write a growth equation for the temperature T. $\Delta T = KT$

b. Write an exponential function giving the temperature T of the culture in terms of minutes passed t.

c. Use the log table to estimate the temperature at the end of 10 minutes.

Find the exponential function corresponding to each of the following tables.

17.

t	y	Δy
0	1	3
1	4	12
2	16	48
3	64	

18.

t	y	Δy
0	2	4
1	6	12
2	18	36
3	54	

19.

t	y	Δy
0	1	.9
1	.1	.09
	.01	.009
	.001	

20.

t	y
0	$\frac{1}{25}$
1	$\frac{1}{5}$
2	1
3	5

21.

t	y
0	1000
1	300
2	90
3	27

22.

t	y
0	3
1	2
2	$\frac{4}{3}$

23.

t	y	Δy
0	400	200
1		300
2		450
3		

24.

t	y	Δy
0	50	-20
1		
2		
3		

Below we have the record of the population of the country of Shrinkia. As you can see they passed right through "zero population growth" (zpg) into negative population growth (npg).

t (years)	y (population)
0	100,000,000 = 10^8
1	99,000,000
2	98,010,000
3	97,029,900
4	96,059,601

25. Write the growth equation $\Delta y = Ky$ for the population.

26. Find the exponential function describing the table.

As a currency inflates, the amount that one unit buys becomes less. In other words, the value of the unit decreases. In order to measure the relative value of a currency, economists pick a given year for a standard. What one unit of currency bought in this year becomes the standard by which following years are measured.

In the country of Autoland the unit of currency is the "caddy." Below we have a record of the value of the caddy over a period of years.

Years	Elapsed time in years (t)	Value of the caddy (y)
1980	0	1.00
1981	1	.80
1982	2	.64
1983	3	.51
1984	4	.41
1985	5	.33
.	.	.
.	.	.
.	.	.

27. In 1981 it takes an entire caddy to buy what a fraction of a caddy (.80 caddy in fact) would have bought a year earlier in 1980.

a. In 1984 one caddy buys what only __(?)__ of a 1980 caddy would have bought. In 1983 one caddy buys what it used to take only .51 of a 1980 caddy to buy. This implies that it takes $\frac{1}{.51}$ or almost two 1983 caddies to buy what one 1980 caddy bought.

b. How many 1985 caddies are needed to buy what one 1980 caddy bought?

28. Write an equation of the form $y = r^t$ which describes the table.

29. a. In Coffeeland an economist noted that the currency lost half of its value in a two-year period. Write an exponential equation which relates the value, y, of a Coffeelandish "bean" (the unit of currency) to time t. Let one time unit equal 2 years.

b. What is the value of the Coffeelandish "bean" after one year?

30. a. Another economist noted that the Coffeelandish bean lost about 30% of its value in one year. Write an equation that relates the rate of change of the bean, Δy, to the value of the bean, y.

b. Write an equation of the form $y = r^t$ that relates the value of the bean, y, to the number of years, t.

An ecologist is studying the amount of water pollutants in a pond. The table on the following page shows the results of his measurements.

31. Convert the table to one in which 1 time unit = 5 days.

32. Find the exponential equation for the table of Problem 31.

t (days)	y (gallons)
0	25
5	20
10	16
15	12.8

SECTION 3: RADIOACTIVITY

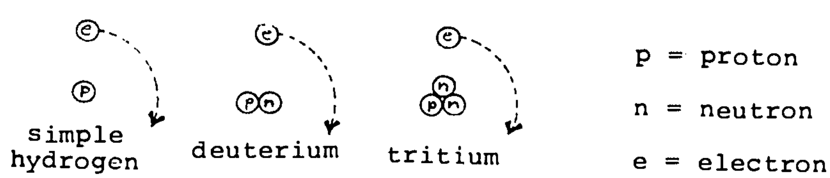
3-1 Isotopes

In Section 1 we stated that exponential functions have many applications in biomedicine. The time has come to present a few such applications. The first subject we will discuss is radioactivity. Radioactive substances are regularly used for killing malignant tissue.

Perhaps you recall from the Respiration Unit that atoms are made up of three kinds of particles. Most of the mass of the atom is concentrated in the nucleus in the form of protons and neutrons. In the usual manner of visualizing the atom, the less massive electrons revolve in orbits around the nucleus. Each electron carries a negative charge and each proton carries an equal positive charge. The neutron is electrically neutral; it carries no charge at all.

In a neutral atom there is no net electrical charge. The number of electrons and protons are equal and consequently the positive and negative charges exactly cancel out. The number of protons in the nucleus is called the atomic number, which determines what element the atom represents. For example, atoms of hydrogen have an atomic number of one. Any atom with one proton in the nucleus is an atom of hydrogen.

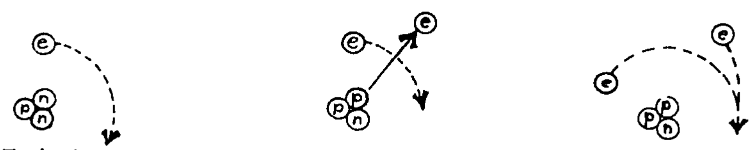
If we add the number of protons and neutrons in the nucleus of an atom we get the mass number of the atom. Sometimes atoms have the same number of protons but different numbers of neutrons, and consequently different mass numbers. Such atoms, all representing the same element but having different mass numbers, are called isotopes of the element. There are three isotopes of hydrogen. They are pictured below.



For short, these isotopes are usually designated as ${}_1H^1$, ${}_1H^2$ and ${}_1H^3$. As you can see, the number of protons is written at the lower left of the H and the mass number at the upper right.

3-2 Radioactive Disintegration

Under most natural conditions, the different isotopes of an element mix together completely. They behave the same way in chemical reactions and indeed, it was a long time before physicists discovered that different isotopes exist. However, there is one way in which some isotopes differ dramatically. In these unstable isotopes, the forces that ordinarily hold atoms together eventually give way in a process called radioactive disintegration. The atom emits some sort of particle or ray and changes into an atom of another element. For example, ${}_1H^3$, the tritium isotope of hydrogen, is unstable. When an atom of tritium disintegrates, a neutron in the nucleus splits into a proton and an electron. The new proton remains in the nucleus and the new electron is ejected away from the atom. The atom now has two protons in the nucleus, but only one revolving electron. In order to become electrically neutral the atom captures a free electron somehow. The final result is ${}_2He^3$, a helium atom. The steps in the process are pictured below.



A Tritium Atom ${}_1H^3$ | Radioactive Disintegration | Electron Capture to form ${}_2He^3$

The above process is only one kind of radioactive disintegration. There are several other kinds, involving the emission of different types of particles.

3-3 The Mathematics of Radioactivity

There is no way to tell exactly when an individual atom of a radioactive substance will disintegrate. If we examine the substance we find that some of the atoms disintegrate after a few seconds and others after days or years. However, when a large number of the unstable atom are present, we can say with certainty what fraction of them will disintegrate within a given length of time called the half-life, at the end of which time half the atoms will have disintegrated. If another half-life period passes, then half of the remaining atoms will disintegrate, and so on. The table below summarizes the process.

Elapsed Time in Half-Lives	Fraction of Substance Remaining
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$ or $(\frac{1}{2})^2$
3	$\frac{1}{8}$ or $(\frac{1}{2})^3$
4	$\frac{1}{16}$ or $(\frac{1}{2})^4$
5	$\frac{1}{32}$ or $(\frac{1}{2})^5$

We can use the table to find an equation for the amount of radioactive substance remaining after any length of time. Suppose that we let A_0 represent the initial amount of the substance, the amount at time zero. If A designates the amount at a later time t, then $\frac{A}{A_0}$ is just the fraction of substance remaining. But the right column of the table says that this fraction is just $(\frac{1}{2})^t$, where t is the time elapsed measured in half-lives and $\frac{1}{2}$ is the common ratio.

$$\frac{A}{A_0} = (\tfrac{1}{2})^t \quad \text{t measured in half-lives}$$

In our usual exponential function notation this becomes

$$A = A_0 \, (\tfrac{1}{2})^t$$

For the purposes of calculation we convert the first of the above equations by taking the log of both sides.

$$\begin{aligned} \log \frac{A}{A_0} &= \log (\tfrac{1}{2})^t \\ &= t \log \tfrac{1}{2} \\ &= t\,(\log 1 - \log 2) \\ &\approx t(0 - .301) \\ &\approx -.301t \end{aligned}$$

EXAMPLE:

$_6C^{15}$, an unstable isotope of carbon, has a half-life of about 2.4 seconds. If you start with 10 grams of this isotope at time zero, how many grams will remain after 3.6 seconds?

SOLUTION:

The initial amount A_0 is given as 10 grams. The time interval represents

$$\frac{3.6}{2.4} = 1.5 \text{ half-lives}$$

You can now use the formula to compute A, the amount remaining.

$$\begin{aligned} \log \frac{A}{A_0} &\approx -.301t \\ \log \frac{A}{A_0} &\approx -.301(1.5) \\ \log \frac{A}{A_0} &\approx -.452 \\ \log \frac{A}{A_0} &\approx .548 - 1 \end{aligned}$$

The mantissa of this logarithm is .548 and the characteristic is -1. Referring to the log table, we find the number closest to .548 in the log y column. It turns out to be .544, which corresponds to a y value of 3.5. Therefore

$$\begin{aligned} \frac{A}{10} &\approx 3.5 \times 10^{-1} \\ &\approx .35 \\ A &\approx 3.5 \text{ gms} \end{aligned}$$

EXAMPLE:

$_4Be^7$, an isotope of Beryllium, has a half-life of about 53.4 days. How many days must pass before the amount of a sample is reduced to 18% of its original value?

SOLUTION:

We know that 18% of the original sample remains. That is,

$$\frac{A}{A_0} \approx .18$$

Substituting into the formula

$$\log \frac{A}{A_0} \approx -.301\ t$$

$$\log .18 \approx -.301\ t$$

we now refer to the log table and find that log .18 = -.745. Therefore

$$-.745 \approx -.301t$$

$$2.5 \approx t$$

Don't forget at this point that t is expressed in half-lives! In order to convert to days, we recall that the half-life is 53.4 days.

$$t(\text{days}) \approx 53.4 \frac{\text{days}}{\text{half-life}} \times 2.5 \text{ half-lives}$$

$$\approx 133.5 \text{ days}$$

PROBLEM SET 3:

1. The atomic number of an atom is the number of __________ in the nucleus.

2. The mass number of an atom is the total number of __________ and __________ in the nucleus.

3. The difference between the mass number and the atomic number gives the number of __________ in the nucleus.

4. The number of __________ determines what element the atom represents.

5. Different isotopes of an element have the same __________ (atomic number, mass number) but different ________________________ (atomic numbers, mass numbers).

6. The __________ of a radioactive isotope is the length of time necessary for the initial amount to be reduced by half.

Problems 7 through 10 should require only a few simple calculations with powers of two. There is no need for a log table.

7. What fraction of a radioactive substance will remain after three half-lives have passed?

8. A certain radioactive isotope has a half-life of one hour. If the initial amount is one gram, how much will remain at the end of six hours?

9. An isotope has a half-life of twenty minutes. Starting with 16 grams, how many hours must pass before only half a gram remains?

10. What is the minimum whole number of half-lives which must pass before less than one-thousandth of a radioactive isotope remains?

In the following problems you will need to make use of the log table.

11. ${}_{8}O^{15}$, an isotope of oxygen, has a half-life of 124 seconds. If the initial amount is 10 milligrams, how much will remain at the end of 297.6 seconds? Round to the nearest .1 milligram.

Not long after the discovery of radioactivity, scientists realized that exposure to radioactive substances can kill human tissue. This opened the way for radiation therapy, which involves the use of radioactive substances to destroy cancerous tissue.

12. The thyroid gland at the base of the neck secretes the hormone thyroxin. This hormone contains iodine. If iodine is injected into the bloodstream, most of it will end up in the thyroid gland. For this reason, injections of the radioactive isotope ${}_{53}I^{132}$ can be used to treat cancerous growths in the thyroid gland.

The half-life of I^{132} is 2.3 hours. What percentage of this isotope will still be present in the body 10 hours after injection? Round your answer to the nearest whole percent.

13. Polycythemia vera is a slow, progressive disease which involves an abnormal increase in red blood cells. One type of treatment involves the injection of the radioactive isotope ${}_{15}P^{32}$ of phosphorus. This isotope tends to accumulate in the cells where blood cells are produced, reducing the output of blood cells.

If the half-life of ${}_{15}P^{32}$ is 14.3 days, what percentage is still around at the end of a week? Round to the nearest whole percent.

14. The potential of radioactive substances to become concentrated in certain tissues can be a serious health hazard. Fallout from nuclear testing is one source of dangerous substances. The isotope ${}_{38}Sr^{90}$ of strontium can be particularly harmful because it behaves much like calcium and builds up in bone tissue and also in human milk.

${}_{38}Sr^{90}$ has a half-life of 28.1 years. Of the strontium deposited in a baby's bones, what percentage will still remain at the end of an average life-time, 70 years? Round to the nearest whole percent.

15. Some radioactive isotopes have half-lives greater than the total age of the earth. One is ${}_{78}Pt^{190}$, an isotope of platinum, with a half-life of 7×10^{11} years. How much time must pass to reduce a quantity of platinum to 1% of its original value?

16. The isotope ${}_{11}Na^{22}$ of sodium has a half-life of 2.6 years. If 8 grams of ${}_{11}Na^{22}$ are present a year after an experiment began, how many grams were present when the experiment began? Round to the nearest tenth of a gram.

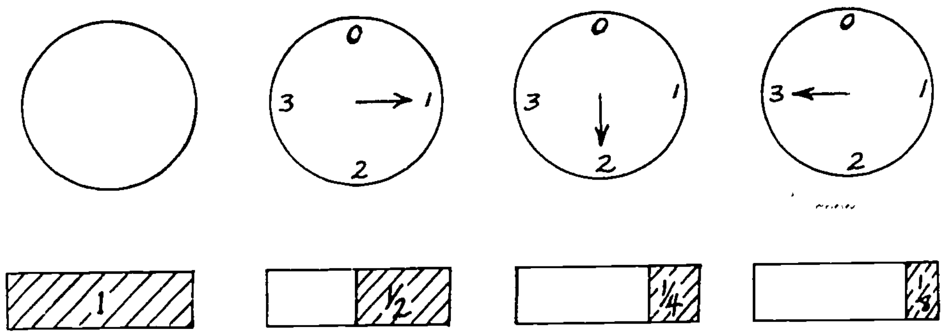
*17. In an experiment using a radioactive isotope, a routine check shows that 60% of the isotope remains. A check one year later shows that 20% remains. What is the half-life of the isotope?

SECTION 4: RADIOACTIVE DATING

4-1 Radioactive Clocks

We can summarize the ideas in the last section by thinking of a radioactive substance as a sort of clock.

Time in Half Lives



Fraction of Isotope Remaining

The face of the clock is calibrated in half-lives, and with each passing half-life the amount of radioactive substance is reduced by half. We can predict the fraction of isotope left after any length of time by using the formula

$$\frac{A}{A_0} = \left(\frac{1}{2}\right)^t \qquad t \text{ in half-lives}$$

Of course the formula could just as well be used in reverse. The decrease in the amount of isotope could be used to compute the elapsed time. This offers the fascinating possibility of determining the ages of ancient artifacts or rocks. If we could measure the present amount of isotope in a rock sample, and then somehow figure out how much of the isotope was present when the rock was formed, we could compute the age of the rock. The theory is simple enough, but the technical problems are formidable. Nevertheless, radioactive isotopes have been used successfully in determining the ages of fossils, rocks and even the earth. Several different isotopes are used; we will concentrate on ${}_6C^{14}$, usually called Carbon-14.

4-2 The Carbon-14 Radioactive Clock

As you know, carbon compounds play a central role in the chemistry of life. All living things contain carbon, which occurs in three isotopes, ${}_6C^{12}$, ${}_6C^{13}$ and ${}_6C^{14}$. Carbon-12 and carbon-13 are stable, but carbon-14 is radioactive. It decays to ${}_7N^{14}$, an isotope of nitrogen. A small percentage of the carbon which exists in the form of carbon dioxide in the atmosphere is carbon-14. Naturally it finds its way into the food chain and consequently living plants and animals are continually taking in small amounts of the radioactive isotope. This pattern stops abruptly once the plant or animal dies. No more carbon-14 is taken in, and that which is present disintegrates as time passes. When we find the remains of a living thing, such as a piece of charcoal from an ancient fire, the amount of remaining carbon-14 is an indication of the time elapsed since its death. Of course the stable carbon-12 and carbon-13 do not decay but simply remain in their original amounts.

Let us look at this process in more detail. In order to pin a specific ·ge on a piece of charcoal, we must know how much carbon-14 there is in it now and how much carbon-14 there was in it when the tree died. We also need to know the half-life of carbon-14. How can these various quantities be determined?

An indispensible tool in all these determinations is the geiger counter. Each time an atom disintegrates it emits a high-velocity particle. When the particle passes through the chamber of the geiger counter, it momentarily closes an electrical circuit and is recorded on the counting mechanism. This leads to a number called the sample activity, which is just the number of disintegrations per minute per gram of sample. If one is lucky enough to know exactly how much radioactive isotope the sample contains, then the sample activity can be used to compute the half-life. The half-life of carbon-14 is usually assumed to be 5568 years, although recent evidence suggests that this figure may be too low.

The next problem centers on determining the amount of carbon-14 still present in the sample of charcoal. How can this be done? Again the geiger counter can be used to advantage. The number of disintegrations is proportional to the amount of carbon-14 present. Therefore one need only measure the sample activity with a geiger counter. This is simple in theory, but not in execution. First, the charcoal may contain other radioactive substances besides carbon-14. These substances would contribute to the geiger count, making the reading too high. This problem can be solved by seperating some carbon from the rest of the sample, which is usually done by burning the sample with oxygen to produce CO_2.

Now the activity of a certain amount of the CO_2 can be measured with a geiger counter. Before this can be done, however, steps must be taken to reduce the background count. Part of this count is caused by cosmic rays which originate outside our solar system. Another part of the background count is due to the small amounts of radioactive substances which contaminate even the most carefully controlled laboratories. In order to cut down on background radiation, the CO_2 sample and geiger counter are placed in a steel cylinder, surrounded by a mercury shield, and placed in an iron vault. This suffices to absorb most foreign particles before they can reach the geiger counter.

After taking the precautions listed above, the activity of the sample can be measured with considerable accuracy. The activity is usually expressed in disintegrations per minute per gram of carbon.

This brings us to the final puzzle. How can we determine the initial amount of carbon-14 in the piece of charcoal? We cannot determine this by direct observation; our answer must be based on guesswork.

The life of a carbon-14 atom starts high in the earth's atmosphere when a nitrogen-14 atom is struck by a cosmic ray. Carbon-14 atoms are continually created this way, and are subsequently mixed throughout the atmosphere. Finally, they decay back to nitrogen-14. Scientists have assumed that the birth and decay processes occur at about the same rate, suggesting that the percentage of carbon-14 in the atmosphere remains the same. Therefore we would expect that the percentage of carbon-14 in living things has not changed much either. When the sample piece of charcoal was part of a tree, we figure that it had the same percentage of carbon-14 as a living tree does today. This means that the charcoal must have originated from a tree which had an activity of 12.6 disintegrations per minute per gram of carbon.

4-3 Mathematics of Radiocarbon Dating

Starting with the formula

$$\frac{A}{A_0} = \left(\frac{1}{2}\right)^t \qquad \text{t in half-lives}$$

it is not hard to develop a specific formula for radiocarbon dating. Since the sample activity is proportional to the amount of carbon-14 present, we can replace the fraction $\frac{A}{A_0}$ by the ratio of sample activities. Let S be the current activity. The original activity was 12.6 disintegrations per minute per gram of carbon. Therefore

$$\frac{A}{A_0} \approx \frac{S}{12.6}$$

So the original formula becomes

$$\frac{S}{12.6} \approx \left(\frac{1}{2}\right)^t \qquad \text{t in half-lives}$$

Notice that t is measured in half-lives. Let x be the time in <u>years</u> and recall that the half-life of carbon-14 is 5568 years. Then

$$x \text{ years} \approx 5568 \frac{\text{years}}{\text{half-life}} \cdot (\text{t half-lives})$$

$$x \approx 5568t$$

$$\frac{x}{5568} \approx t$$

Substituting this expression in for t in the formula gives

$$\frac{S}{12.6} \approx \left(\frac{1}{2}\right)^{\frac{x}{5568}} \qquad \text{x in years}$$

Now we take the common logarithm of each side

$$\log \frac{S}{12.6} \approx \log \left(\frac{1}{2}\right)^{\frac{x}{5568}}$$

$$\log S - \log 12.6 \approx \frac{x}{5568}(\log 1 - \log 2)$$

$$\approx \frac{x}{5568}(0 - .301)$$

Referring to a log table we find that log 12.6 = 1.1. Therefore

$$\log S - 1.1 \approx -\frac{.301x}{5568}$$

Solving this equation for x, we get

$$x \approx 20{,}350 - 18{,}500 \cdot \log S$$

EXAMPLE:

Suppose that a piece of charcoal has an activity of 10.0 disintegrations per minute per gram of carbon. What is its age?

SOLUTION:

$S = 10.0$ which means that $\log S = 1$. Therefore

$$x \approx 20{,}350 - 18{,}500 \cdot 1$$

$$\approx 1850 \text{ years}$$

4-4 Final Remarks on Radiocarbon Dating

We mentioned earlier that radiocarbon dating requires certain assumptions. The primary assumption is that the proportion of carbon-14 in the atmosphere, and consequently in living things, has been constant for the past 40,000 years. Recently it has been possible to test this assumption through the study of tree growth rings. Each year a tree forms a new growth ring. In order to determine the age of a tree one need only take a section and count the rings. rtain bristlecone pine trees from the White Mountains of California have been fo to be as much as 4600 years old. Using a combination of living and dead trees, ntists have managed to trace tree rings back for about 8000 years.

How does this tie in with radiocarbon dating? The important point is that any inner ring of a living tree is essentially dead; it is no longer incorporating new carbon. Therefore, the carbon-14 in the ring has been decaying since the year the ring was formed. This means that the tree ring can be dated by the radiocarbon method and the result compared with the true age. This technique has shown that some radiocarbon dates are in error by as much as 700 years. Apparently the incidence of carbon-14 in the atmosphere has varied somewhat, probably because of different cosmic ray levels in our upper atmosphere. Radiocarbon dating is still very useful, but the values so obtained have to be corrected.

PROBLEM SET 4:

1. Carbon-14 atoms are produced through bombardment of nitrogen-14 atoms by __________.

2. The half-life of carbon-14 is usually assumed to be ______________ years.

3. The sample activity of living trees today is about _____________ disintegrations per minute per gram of carbon.

The following numbers are typical sample activities measured in disintegrations per minute per gram of carbon. In each case determine what percentage of the original carbon-14 remains in the object.

4. 6.3 for a piece of shell

5. 2.52 for a piece of rope

6. 4.41 for a mummified aardvark

The noted archeologist Professor Arte Fact has just finished measuring the sample activities of a few relics lying around his laboratory. All the activities are measured in disintegrations per minute per gram of carbon. Find the age of each object. You will need to use the log table in these problems.

7. 11.0 for wood from an old log

8. 9.5 for wood from a mummy coffin

9. 9.2 for wood from a Syrian palace

10. 8.7 for part of a Sequoia tree felled in 1874

11. 8.0 for a piece of an Egyptian funeral boat

12. Uranium-238 has a half-life of 4.5×10^9 years. A scientist reasons that a certain rock contains about 62% of the uranium-238 which it contained when the earth first cooled from a molten state. Use this information to estimate the age of the earth.

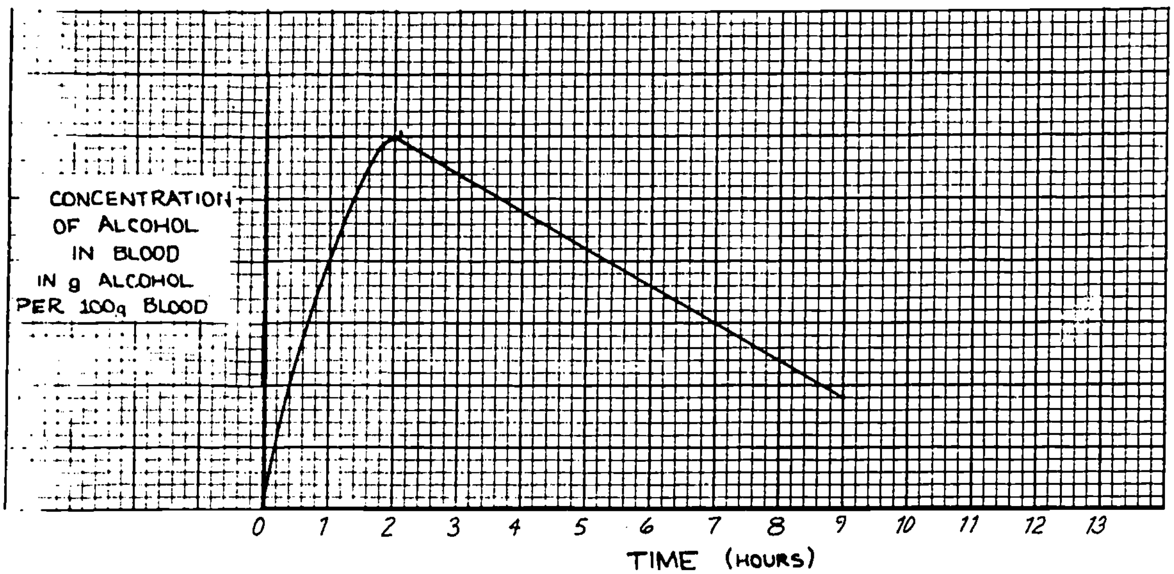
SECTION 5: DRUG METABOLISM AND ELIMINATION

5-1 Drug Concentrations in the Bloodstream

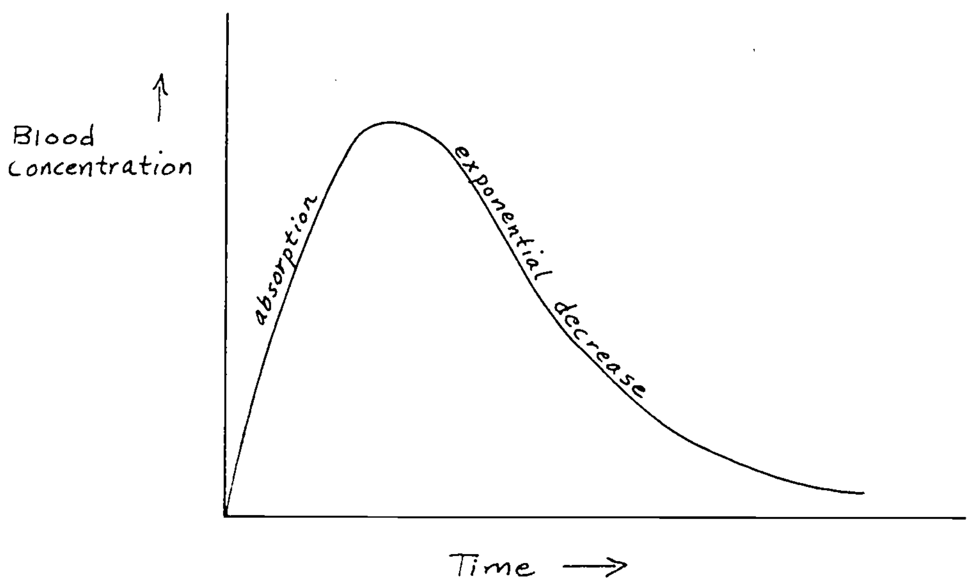
What might we expect to observe when a drug is given to a patient? First the drug is absorbed and distributed throughout some part of the body. There will be parts of the body where the drug is not absorbed, bones for example. On the other hand almost all drugs will distribute themselves throughout the bloodstream. The exact volume of the body which will absorb the drug will vary depending on the drug and the size of the individual. This volume is called the volume of distribution.

As the drug is absorbed, the concentration in the bloodstream will rise until a peak level is reached. Then the concentration will begin to drop as the drug is excreted in the urine and broken down by the metabolic processes of the body. This is the phase of metabolism and elimination.

You have already studied the above processes in the case of alcohol. The graph on the following page shows blood concentration as a function of time. For the first two hours the alcohol is being absorbed and the blood concentration is rising. Then it reaches a peak and starts to decrease.



Notice that the metabolism and elimination phase is described by a linear function. In this respect alcohol is very unusual. Most drugs obey a law of exponential decrease in this phase. This is true because the rate at which the drug is metabolized (and eliminated) is proportional to the amount of drug present in the body. Whenever this type of proportionality occurs, an exponential function will turn up. We can describe the situation for most drugs by the following curve.



The decrease in the blood concentration of the drug is similar to that of a radioactive isotope. Different drugs have different <u>half-lives</u>, just as in the case of isotopes. A knowledge of the half-life of a drug allows us to predict future blood levels and therefore to plan the proper interval between doses. The study of drug absorption, metabolism and elimination is called <u>pharmacokinetics</u>. This is a very active area of research today.

5-2 Calculations for a Specific Example

In order to do some calculations we are going to make the simplifying assumption that the absorption phase is instantaneous. That is, we will assume that the drug distributes through the volume of distribution immediately. In the case of an intravenous injection this assumption is not too far off base.

Suppose we take the drug Digoxin, which is used for heart patients. Suppose that the initial dose is .5 mg, the volume of distribution is 400 liters and the half-life is 36 hours.

EXAMPLE:

Find the initial blood concentration in $\frac{mg}{liter}$.

SOLUTION:

In this case the volume of distribution is too large to be considered as an actual volume in the body, but we proceed as if it were. In this case the volume of distribution behaves more as a proportionality constant.

We have .5 mg of the drug distributed over 400 liters. Therefore the initial concentration is

$$\frac{.5 \text{ mg}}{400 \text{ liters}} = 1.25 \times 10^{-3} \ \frac{mg}{liter}$$

EXAMPLE:

Find the concentration of the drug at the end of 24 hours.

SOLUTION:

We may use the same equations which applied to radioactive isotopes. If A_0 is the initial concentration of the drug, A the new concentration and t is measured in half-lives, then

$$A = A_0 \ (\tfrac{1}{2})^t$$

or

$$\log \frac{A}{A_0} \simeq -.301t$$

Now 24 hours represents $\frac{24}{36} \simeq .67$ half-lives. Substituting t = .67 we have

$$\log \frac{A}{A_0} \simeq -.301 \ (.67)$$

$$\simeq -.202$$

Hence $\frac{A}{A_0} \simeq .63$

Therefore

$$A \simeq (.63)A_0$$

$$A \simeq (.63)(1.25 \times 10^{-3})$$

$$A \simeq 7.9 \times 10^{-4} \ \frac{mg}{liter}$$

EXAMPLE:

At the end of 24 hours the patient receives an additional dose of .16 mg. What is the new blood concentration?

SOLUTION:

According to the above calculations the patient has a concentration of $7.9 \times 10^{-4} \frac{mg}{liter}$ when the dose is given. The new dose of .16 mg distributes over the volume of distribution of 400 liters, giving an additional concentration of

$$\frac{.16 \text{ mg}}{400 \text{ liters}} = 4 \times 10^{-4} \frac{mg}{liters}$$

In order to obtain the new concentration we must add concentration due to the new dose and the concentration remaining from the initial dose.

$$4 \times 10^{-4} \frac{mg}{liter} + 7.9 \times 10^{-4} \frac{mg}{liter} = 1.19 \times 10^{-3} \frac{mg}{liter}$$

PROBLEM SET 5:

1. Merperidine (demerol) is a drug used for the relief of pain. The volume of distribution is 50 liters and the half-life is 3 hours. No log table is needed to answer the following questions. Round answers to the nearest .01 $\frac{mg}{liter}$.

a. Suppose that an initial dose of 50 mg is given. What is the initial concentration in $\frac{mg}{liter}$?

b. What will the concentration be at the end of 6 hours?

c. Suppose that at the end of 6 hours, an additional 30 mg is given. What will the new concentration be?

d. What will the concentration be at the end of another 6 hour period?

e. If another dose of 30 mg is then given, what will the concentration be?

2. The antibiotic Erythromycin has a volume of distribution (for a typical person) of 40 liters and a half-life of 1.5 hours. No log table is needed to answer the following questions. Round answers to the nearest .1 mg/liter.

a. If an initial dose of 400 mg is given what is the initial blood concentration in mg/liter?

b. At the end of 6 hours what is the concentration?

c. At the end of 6 hours an additional 400 mg is given. What is the new concentration?

d. What will the concentration be at the end of another 6 hour period?

e. If an additional dose of 400 mg is then given, what will the new concentration be?

3. The drug Quinidine is often used in cases of heart failure. The half-life is 6 hours and the volume of distribution is 16 liters. You will need to refer to the log table in answering the following questions. Round to the nearest 1 $\frac{mg}{liter}$.

a. If an initial dose of 400 mg is given what is the initial blood concentration?

b. At the end of 8 hours, what will the concentration be?

c. If an additional dose of 400 mg is then given what will the new concentration be?

d. At the end of another 8 hour period what will the concentration be?

The antibiotic Streptomycin has a volume of distribution of 20 liters. The half-life is 2.5 hours.

4. An initial dose of 500 mg is given. How long will it be before the blood level drops to 4.75 $\frac{mg}{liter}$?

5. An initial dose is given and at the end of 5 hours the concentration is 5 $\frac{mg}{liter}$. What was the size of the initial dose in mg?

The antibiotic Penicillin G has a volume of distribution of 30 liters and a half-life of .7 hours.

6. A dose of 450 mg is given. What is the initial concentration?

7. How long before the concentration has dropped to 1 $\frac{mg}{liter}$ (to nearest .1 hr)?

8. How long before the concentration drops to .01 $\frac{mg}{liter}$ (to nearest .1 hr)?

SECTION 6: EPIDEMICS

6-1 Examples of Epidemics

The word epidemic is used to describe a situation in which a large portion of a population is stricken by a single disease. Disastrous epidemics have occurred intermittently throughout history and at times, the resulting deaths have significantly reduced the total human population. In order to occur in epidemic proportions, a disease must be highly contagious, and the immunity of the population must be relatively low. Naturally, only certain diseases can create epidemics, and it is no surprise that we encounter the same ones repeatedly in historical accounts.

Among all the diseases which have occurred in epidemic proportions, the most devastating has been bubonic plague, or "black death." Bubonic plague is primarily a disease in rats, but it is sometimes carried to humans when they are bitten by infected fleas. Wherever humans and rats live in close association, as in most cities, there is the danger of plague infection. In humans, the disease attacks the lymph and respiratory systems, is highly contagious and spreads rapidly. Before the development of adequate medical care, 50 to 90 percent of cases ended in death.

Early writings confirm that bubonic plague was epidemic long before the Christian era. Many outbreaks have occurred since then. A severe epidemic between 1347 and 1351 killed 75 million people. In London, during the early seventeenth century, up to four thousand people a week died of plague. The last major epidemic occurred in the late 1800's and spread worldwide.

Although the plague bacteria is permanently established in rodent populations in the United States, the disease has not been an important problem in this century. Antibiotics such as Streptomycin are effective in treatment and vaccines are available for protection against epidemics.

Another epidemic disease which does not respond to antibiotics and which has remained a problem in the twentieth century is influenza. Influenza is a virus infection of the respiratory system, often accompanied by complications such as pneumonia. It began to appear in epidemic form about a century ago, and in 1918, it was responsible for between 20 and 40 million deaths. More recently in 1957-1958, 40 million people in the United States became ill with a variety of influenza called Asian flu. Authorities estimate that more than 60 thousand deaths resulted from Asian flu and resulting medical complications during this epidemic.

The only protection against influenza is to be found in periodic vaccination. If a person does become ill with the disease the most important factor is prompt treatment of any resulting medical complications.

6-2 The Growth of an Epidemic

The typical epidemic begins with only a few cases of the illness and then grows rapidly as more people are infected. The general pattern is clear from the following table which gives the weekly plague deaths in London during part of the year 1625.

Week ending		Plague deaths
April	7	10
"	14	24
"	21	25
"	28	26
May	5	30
"	12	45
"	19	71
"	26	78
June	2	69
"	9	91
"	16	165
"	23	239
"	30	390

The growth is erratic but in general it seems that the more people affected, the faster the epidemic grows. Following this line of reasoning we might guess that the rate of infection of an epidemic is proportional to the number of people already infected. If we let I represent the number of infected individuals, this notion can be expressed as follows.

$$\Delta I = KI$$

Therefore I must be an exponential function of time.

$$I = Ar^{t}$$

This equation might be fairly accurate at the beginning of an epidemic but what about later on? No epidemic can grow indefinitely because eventually there are no more people to catch the disease. Therefore the growth of the epidemic must slow down at some point. The equation above provides a promising start but it fails in this respect because it never stops growing.

6-3 Another Approach

In order to come up with a more accurate model we must rethink the problem. Suppose that the following conditions hold for a certain disease.

1. Everyone is susceptible to the disease (i.e., no one is immune)
2. Once infected, an individual is always infected and can spread the disease.
3. There are no deaths from the disease.
4. No measures are taken to prevent the spread of the infection (i.e., there is no medical treatment, etc).

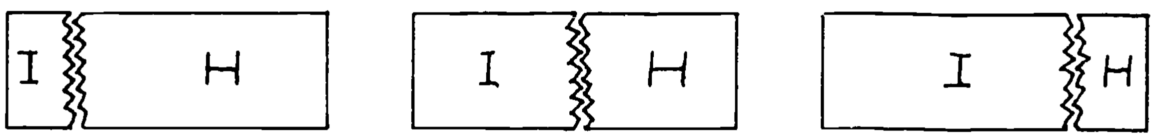
Suppose we concentrate on the population of a given city. Let

N = total population of the city
H = number of healthy individuals
I = number of infected individuals

Note that at any time the total N is just the sum of the numbers of healthy and infected individuals.

$$N = H + I$$

How would we expect the number I to grow? If I is small then the infection will not grow too fast. Likewise, if H is small then I will not grow fast either because there are only a few people left to contact the disease. I will grow fastest when both I and H are large.



Slow Growth   Fast Growth   Slow Growth

It turns out that the following equation fits the situation we are describing.

$$\frac{H}{I} = Na^t$$

where

a = some constant characteristic of the disease
t = time measured in days

$\frac{H}{I}$ is the ratio of healthy to infected individuals; suppose we call this ratio R. The number of infected people can be easily computed.

$$\frac{H}{I} = R$$

Remembering that H + I = N, the above equation can be written as

$$\frac{N - I}{I} = R$$

Solving for I,

$$N - I = IR$$

$$N = I + IR$$

$$= I(1 + R)$$

Therefore

$$I = \frac{N}{1 + R}$$

PROBLEM SET 6:

1. Prepare a graph displaying the data on Plague deaths shown in the table of section 6-2.

Suppose that an epidemic in New York City begins when one person becomes infected with a disease. The population of New York City is about 10 million, so $N = 10^7$. Suppose that $a = \frac{1}{10}$ for the disease under consideration. Then the equation

$$\frac{H}{I} = Na^t$$

becomes

$$\frac{H}{I} = 10^7 \times (\frac{1}{10})^t$$

2. Fill in the second column of the table shown opposite.

t in days	$\frac{H}{I} = R$	I Number of Infected Persons
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		

The equation

$$I = \frac{1}{1 + R} \cdot N$$

for the number of infected people, now becomes

$$I = \frac{1}{1 + R} \cdot 10^7$$

3. Fill in the third column of the table. Round all of your entries so that there is only one non-zero digit.

4. On a peice of graph paper plot I as a function of t in days.

5. Write an equation of the form $I = Ar^t$ which describes the course of the epidemic for the days 0 to 5.

SECTION 7: CONVERTING EQUATIONS INTO THE FORM $y = Ar^t$

7-1 Introduction

You have now seen several examples of the application of exponential functions in applied problems. Two aspects have been considered so far. One involves making predictions when an exponential function is given, such as predicting the course of an epidemic or the remaining amount of a radioactive isotope. Secondly, you have seen how to find the exponential function corresponding to a table of values. However, the tables were of a special type. First, the time values were always evenly spaced and second, there was always an exponential function which fit the given values exactly.

If we are dealing with a table of data from an actual experiment the chances are remote that an exponential function will fit it exactly. This is because error is involved. In the coming sections we will endeavor to answer two questions.

1. Given a data set, how can we determine if an exponential function is appropriate to describe it?

2. What is the equation of the "best" exponential function describing a given set of data?

7-2 Converting Equations into the Form $y = Ar^t$

In order to answer the questions above it will be necessary to work with exponential functions which appear in various forms. Up to this point all the functions have been expressed in the form $y = Ar^t$. In order to convert equations into this form we must make use of the following basic exponential properties.

$$(a^s)(a^w) = a^{s+w}$$

$$(a^s)^w = a^{sw}$$

$$a^{-s} = \frac{1}{a^s}$$

We will also need to recall that the functions $y = 10^x$ and $x = \log y$ are inverse functions. Therefore the following identities hold.

$$10^{\log y} = y$$

$$\log 10^x = x$$

We are now ready to look at some examples of converting equations into the form $y = Ar^t$.

EXAMPLE:

Convert the euqation

$$y = 5(2)^{6t}$$

into the form $y = Ar^t$.

SOLUTION:

Notice that the variable t has a coefficient of 6. It is for this reason that the function is not in the form $y = Ar^t$.

We use the multiplicative law of exponents.

$$y = 5(2)^{6t}$$

$$= 5(2^6)^t$$

$$= 5(64)^t$$

The equation is now in the desired form with A = 5 and r = 32.

EXAMPLE:

Convert the equation $y = 100(3)^{-2t}$

SOLUTION:

We again use the multiplicative law along with the property $a^{-s} = \frac{1}{a^s}$.

$$y = 100(3)^{-2t}$$
$$= 100(3^{-2})^t$$
$$= 100\left(\frac{1}{3^2}\right)^t$$
$$= 100\left(\frac{1}{9}\right)^t$$

Hence, $A = 100$ and $r = \frac{1}{9}$.

EXAMPLE:

Convert the equation $y = 2^{4x + 7}$

SOLUTION:

In this example the additive law of exponents must also be used.

$$y = 2^{4x + 7}$$
$$= (2^{4x})(2^7)$$
$$= (2^7)(2^{4x})$$
$$= 128\ (2^{4x})$$
$$= 128\ (2^4)^x$$
$$= 128\ (16)^x$$

Hence $A = 128$ and $r = 16$.

EXAMPLE:

Convert the equation $\log y = 2t - 3$

SOLUTION:

In order to eliminate the log function on the left side we recall that $10^{\log y} = y$. We therefore write each side of the equation as a power and then proceed as before.

$$10^{\log y} = 10^{2t-3}$$
$$y = 10^{2t-3}$$
$$y = (10^{2t})(10^{-3})$$
$$y = (10^{-3})(10^{2t})$$
$$y = \frac{1}{10^3}\ (10^2)^t$$
$$y = \frac{1}{1000}(100)^t$$

Often it is necessary to perform calculations with logarithms in problems of this type, as the following example illustrates.

EXAMPLE:

Convert the equation $\log y = -.3t + 1.2$

SOLUTION:

Again we write both sides of the equation as powers and use the fact that $10^{\log y} = y$.

$$10^{\log y} = 10^{-.3t+1.2}$$

$$y = 10^{1.2}\,(10^{-.3})^t$$

$$y = 10^{1.2}\,\left(\frac{1}{10^{.3}}\right)^t$$

The log table is now used to determine $10^{1.2}$ and $10^{.3}$. We find that $10^{1.2} \approx 16$ and $10^{.3} \approx 2$. Therefore

$$y \approx 16\left(\tfrac{1}{2}\right)^t$$

<u>EXAMPLE</u>:

In the presence of adequate nutrients the bacteria <u>E</u>. <u>Coli</u> will double each 17 minutes. Starting with a population of one cell at time zero, write an equation relating the population y to the time t measured in <u>hours</u>.

<u>SOLUTION</u>:

We can begin by noting that if S represents time measured in 17 minute units then the equation must be

$$y = 2^S$$

Now let t represent time measured in hours. When t = 1, S must be $\frac{60}{17} \approx 3.53$. Therefore $S \approx 3.53t$. Hence

$$y \approx 2^{3.53t}$$

is the desired equation.

Finally we convert this to the form $y = Ar^t$.

$$y \approx (2^{3.53})^t$$

Using the logarithm table we find that $2^{3.53} \approx 11.5$. Therefore

$$y \approx (11.5)^t$$

PROBLEM SET 7:

Convert each of the equations in Problems 1 - 23 to the form $y = Ar^t$ or $y = Ar^x$.

1. $y = 3^2t$
2. $y = 2^{4x}$
3. $y = 5^{3t}$
4. $y = 3(2^{3t})$
5. $y = 100(7^{2x})$
6. $y = 50\left(\frac{1}{3}\right)^{3x}$
7. $y = 1000(.1)^{3t}$
8. $y = 8^{-2x}$
9. $y = 10^{-3x}$
10. $y = 12^{-t}$
11. $y = 2^{-5t}$
12. $y = 8^{t+2}$
13. $y = 5^{2x-2}$
14. $y = 10^{-2t+3}$
15. $y = 3(2^{-4x+3})$
16. $\log y = 3t + 2$
17. $\log y = -2t + 4$
18. $\log y = -x - 3$
19. $\log y = 6t - 2$
20. $\log y = .5x + .2$
21. $\log y = .3t - 3.4$
22. $\log y = -.4x + 1.3$
23. $\log y = -1.6t - 2.8$

Solve for y.

*24. $\log y = 9 \log t + 2.2$

In each of the following problems a type of bacteria is given along with the length of time necessary for the population to double. In each case write an equation describing the population as a function of time <u>measured in the units given</u>. Assume each population starts with one cell.

25. <u>Bacillus</u> <u>mycoides</u> 28 min., t in hours. (Round the base to nearest .1)

26. <u>Bacillus</u> <u>thermophilus</u> 18 min., t in hours. (Round base to nearest integer)

27. <u>Lactobacillus</u> <u>acidophilus</u> 75 min., t in hours. (Round base to nearest .01)

28. <u>Mycobacterium</u> <u>tuberculosis</u> 900 min., t in days. (Round base to nearest .01)

29. <u>Rhizobium</u> <u>japonicum</u> 360 min., t in days. (Round base to nearest integer)

30. <u>Treponema</u> <u>pallidum</u> 1980 min., t in days. (Round base to nearest .001)

SECTION 8: FINDING THE EQUATION FOR A SET OF DATA

In Section 7, two questions were posed. We repeat them here.

1. Given a data set, how can we determine whether an exponential function is appropriate to describe it?

2. What is the equation of the "best" exponential function describing a given set of data?

In this section we will present a procedure for answering both these questions. The seeds of this technique appeared in Problem Set 7 in which you were asked to convert equations such as

$$\log y = 5x + 3$$

into the form $y = Ar^x$. In other words, you were asked to convert a linear relationship between log y and x into an exponential relationship between y and x.

This process also works in reverse; if y and x are related by an exponential function

$$y = Ar^x$$

then we can take the log of both sides of the equation to obtain a linear relationship between log y and x.

$$y = Ar^x$$

$$\log y = \log Ar^x$$

$$\log y = (\log r)x + \log A$$

We are now in a position to answer the first question above.

<u>If the relationship between log y and x is linear then the relationship between y and x is exponential.</u>

As an example, suppose we wish to know if an exponential function will describe the relationship between concentration and per cent transmittance in the table below. This table records sample results from Laboratory Activity 31 in Biomedical Science, Unit I.

Concentration x ml indicator in 10 ml solution	Per Cent Transmittance y
0	100.0
1	89.5
2	68.0
3	59.0
4	47.5
5	37.0
6	30.5
7	24.0
8	19.0
9	14.0
10	11.5

Step 1:

Add a log y column to the table.

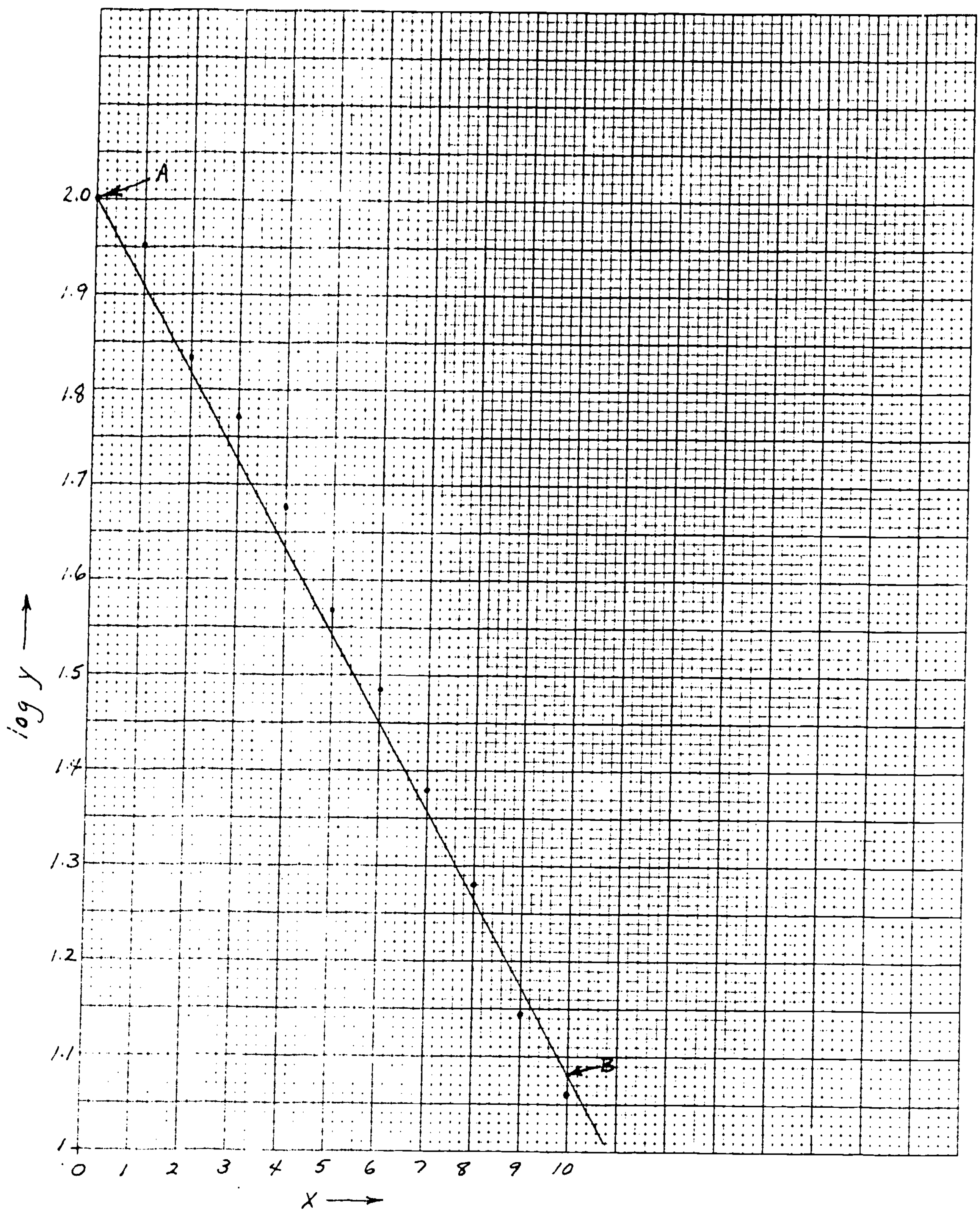
x	y	log y
0	100.0	2.000
1	89.5	1.952
2	68.0	1.833
3	59.0	1.771
4	47.5	1.677
5	37.0	1.568
6	30.5	1.484
7	24.0	1.380
8	19.0	1.279
9	14.0	1.146
10	11.5	1.061

Step 2:

Graph the (x, log y) ordered pairs as shown on the following page.

A line has been drawn that best approximates the point set. The relationship between log y and x is very nearly linear, and therefore the relationship between y and x is very nearly exponential.

Once we have decided that an exponential function is appropriate, the final step is to find the equation for the function.



A
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
B
0
1
2
3
4
5
6
7
8
9
10
log y
X

Step 3:

Draw the best approximating line and find its equation. The line has already been drawn on the graph. In order to find the equation of the line, we need to determine its slope and log y - intercept. The log y-intercept is easily seen to be 2 by inspecting the graph. The slope is determined in the usual manner. We determine the rise and run from the two points A and B.

$$A = (0,2) \qquad B = (10,1.08)$$

$$\text{rise} = 1.08 - 2$$

$$= -.92$$

$$\text{run} = 10 - 0$$

$$= 0$$

$$\text{slope} = \frac{-.92}{10}$$

$$= -.092$$

We now have the information we need to write an equation for the line.

$$\log y \approx -.092x + 2$$

Step 4:

Convert the linear equation to an exponential equation. In order to do this you simply follow the same technique as in the last problem set.

$$10^{\log y} \approx 10^{-.092x+2}$$

$$y \approx (10^2)(10^{-.092})^x$$

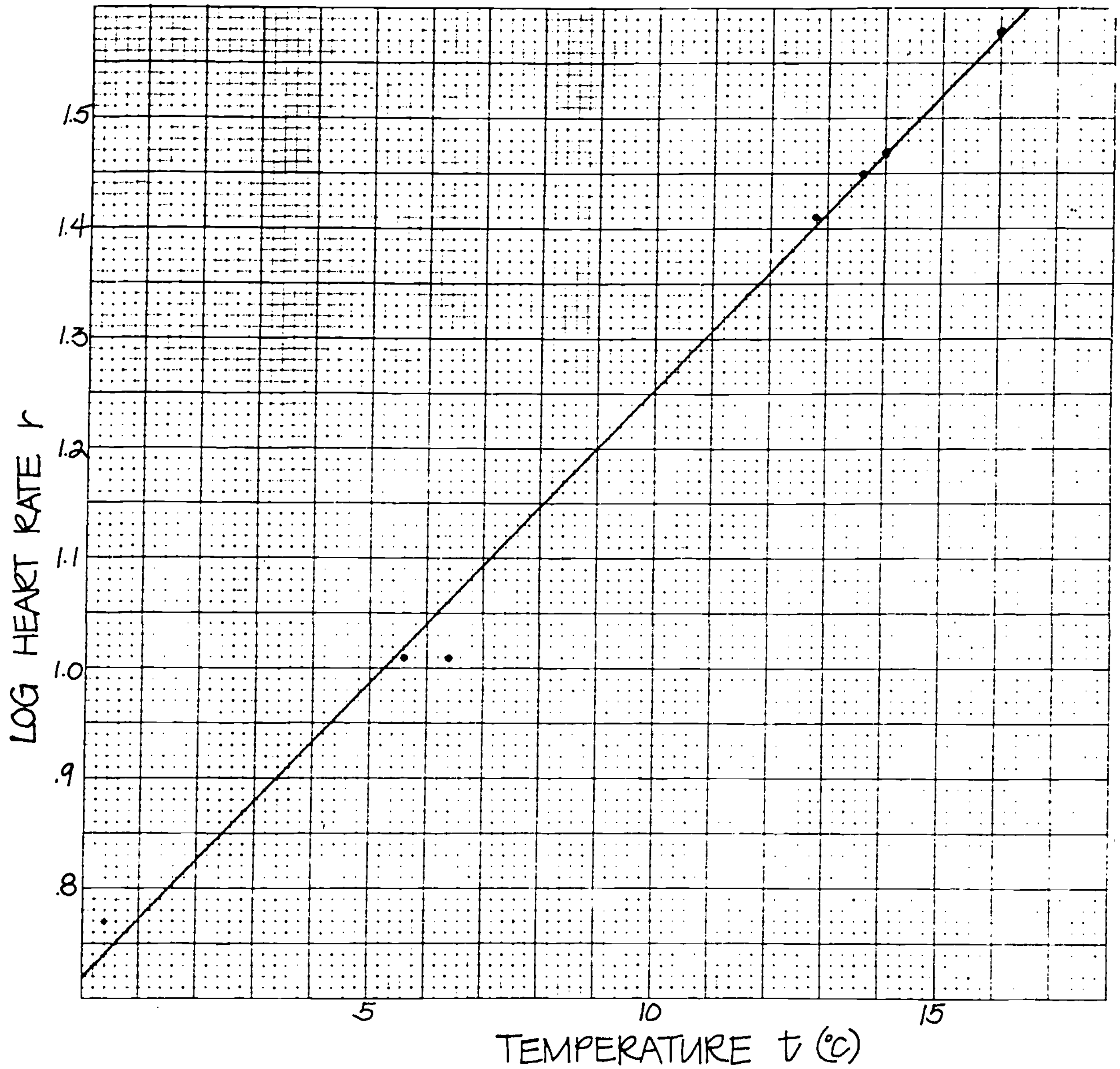
$$y \approx 100(.81)^x$$

and we are done.

The technique just described allows us to start with a table of values and end up with an equation. It is important to note that it also allows us to see the degree to which error enters into the process. The role of error may be seen in the degree of nonlinearity of the points. For example, suppose the points looked like a flyswarm. Then the effect of error would be so bad as to defy description. Certainly an exponential equation wouldn't describe the data. On the other hand suppose the points looked like they were on a smooth curve. Then, we would have reason to suspect that an exponential equation was inappropriate.

PROBLEM SET 8:

The graph on the following page shows the results of several measurements of a rabbit's pulse rate r as a function of atmospheric temperature t in degrees Celcius. As you can see, log r has been plotted as a function of t and an approximating line drawn in.



1. What is the vertical intercept of the line? to the nearest .01?

2. What is the slope of the line? Round to nearest .001.

3. Write the approximate equation of the line.

4. Find the exponential equation corresponding to the equation of Problem 3.

Follow the procedure in the text to write an equation of the form

$$y = Ar^x$$

which approximately describes the data in each table shown on the following page. Show all work.

42

5.

x	y
0	99.5
1	78.0
2	64.0
3	46.5
4	36.5
5	29.5
6	22.0
7	16.5
8	14.0
9	11.0
10	8.0

6.

x	y
0	100.0
1	80.0
2	67.0
3	54.5
4	46.5
5	38.0
6	33.0
7	28.5
8	23.5
9	19.5
10	16.5

On the following page is a picture of a seashell which has been sliced in half. The shell belongs to an animal called the chambered nautilus which lives in deep waters of the Pacific and Indian oceans. The shell resembles that of a snail and is divided into numerous compartments. At one time or another, the animal lived in each compartment, moving to the next larger one when quarters got cramped.

It turns out that the nautilus shell grows at a rate proportional to its size, which leads to an exponential relationship. The relationship can most easily be studied by looking at the edge of the shell which spirals outward from the center.

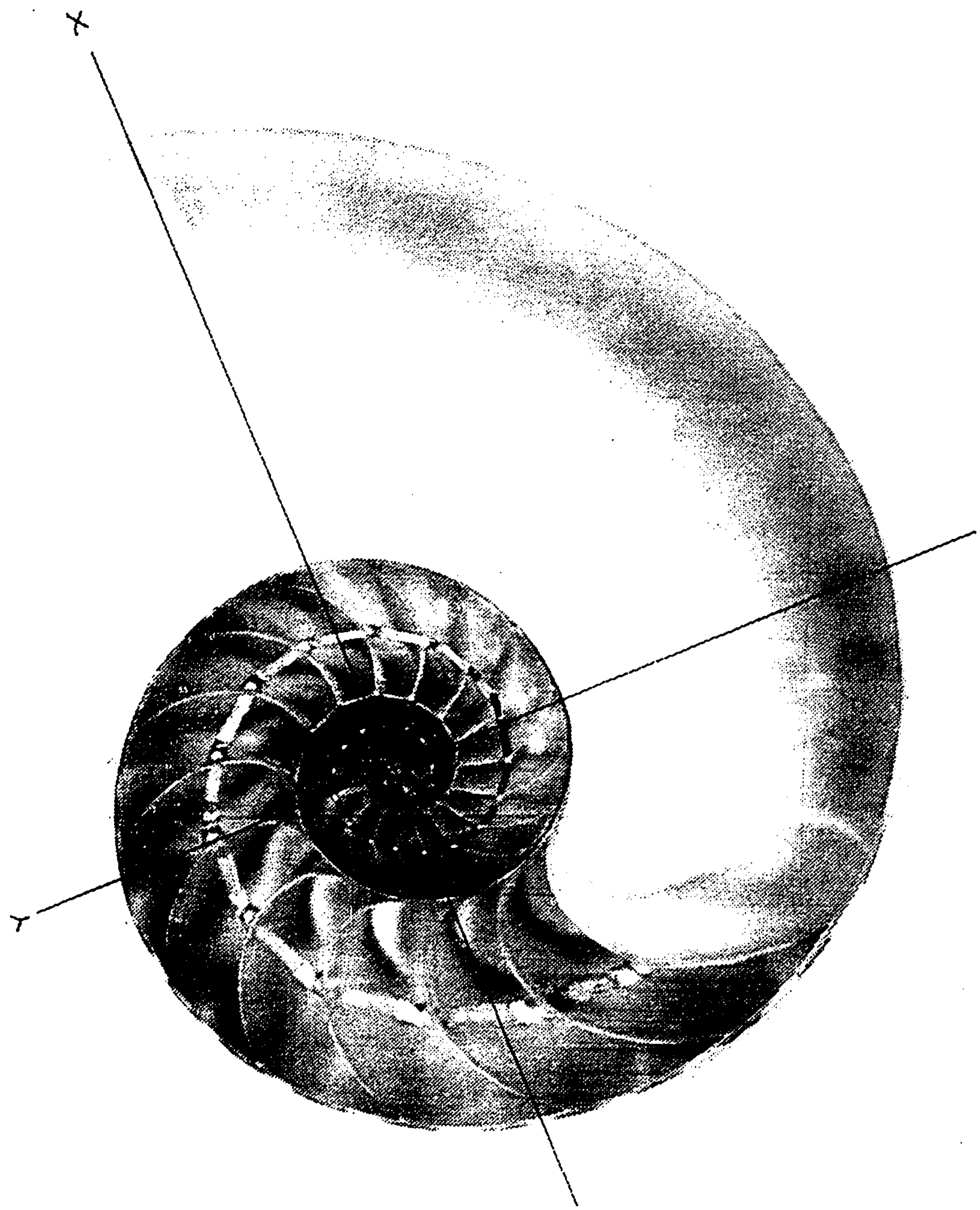
Your instructor will give you detailed instructions on the following problems.

7. Complete the r and log r columns in the table shown opposite. r is the distance from the origin of the coordinate system to the edge of the spiral (measure in centimeters). The θ entries are integral multiples of 90°.

8. On a piece of graph paper, plot log r as a funciton of θ (in radians) and draw the best line through the resulting points.

9. Find the exponential equation relating r and θ.

θ	r (cm)	log r
90°		
180° (π)		
270°		
360° (2π)		
450°		
540° (3π)		
630°		
720° (4π)		
810°		
900° (5π)		
990°		
1080° (6π)		



SECTION 9: A COOLING EXPERIMENT

9-1 An Example of an Exponential Function

In this section you are going to perform a simple experiment that leads to a concrete example of an exponential function. In the process you will use graphing techniques to find the equation for the specific exponential function.

Suppose that a volume of water is heated to boiling and then removed from the heat. What will happen to the temperature of the water? Certainly the water will cool, approaching room temperature. If we let D be the difference between the water temperature and room temperature, then we expect D to get smaller as time passes. Moreover, it seems reasonable that when D is large, the cooling will be more rapid than when D is small. The larger D is, the faster D can be expected to change. This suggests that the rate of change of D will be proportional to D.

$$\Delta D = kD$$

for some proportionality constant k.

Whenever the rate of change of a thing is proportional to that thing itself, we are dealing with an exponential function. In the above example we expect an exponential function of the form

$$D = Ar^t$$

Where t is time and A and r are constants. Of course this equation will hold only if we are correct in assuming that the rate of change of D is proportional to D.

9-2 Instructions for the Experiment

The previous assumptions can be investigated by actually conducting a cooling experiment. Your teacher will assign you to small groups for this purpose. At the beginning of the experiment your group should have the following equipment.

1. one styrofoam cup
2. one Celcius thermometer which can register temperatures up to 100° C.
3. a stand for securing the thermometer
4. pencil and paper
5. a watch with a second hand or a wall clock having a second hand within sight

You are now ready to conduct the experiment. The following instructions should be followed carefully.

1. Use the stand to mount the thermometer so that the bulb is touching the bottom of the styrofoam cup (see Figure 1).

2. Prepare a table like that on Page 41 of the text. Fill in the time column exactly as in that table. Leave the other columns blank.

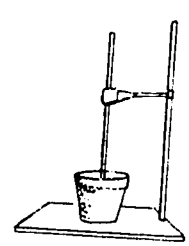


FIGURE 1

3. When you are convinced that the thermometer reading has stabilized, record the temperature to the <u>nearest half degree</u> at the top of your table and label it <u>room temperature</u> (see Page 41 of text).

4. Ask the teacher to place 20 ml of hot water in your styrofoam cup.

5. Watch the thermometer. When the temperature has reached its peak and just starts to descend, note the time and record the temperature to the <u>nearest half degree</u> to the right of time 0:0 in the column labeled T (°C).

6. Continue making temperature readings, always to the <u>nearest half degree</u>. You will find it most efficient to have one person read the temperature, one to keep track of the time and one to record the results. It is <u>important</u> to notice that temperatures are recorded every half minute for the first 6 minutes and every full minute after that.

9-3 Analysis of Results

The next step is the analysis of your experimental results. The purpose is to check if the cooling function is exponential, and if so, to find the equation of the function. We have hypothesized that

$$D = Ar^t$$

Where A and r are constants. We can use the method of Section 8 to see if this relationship holds. That is, we graph log D as a function of t and see if a straight line results.

All the necessary calculations are done below on sample data collected by Biomed student Ann Chovie. You should work with your group in doing the same calculations on your own data.

As a first step Ann Chovie completed her data sheet, filling in the D and log D columns (see page 41). Each value of D is obtained by subtracting room temperature (26° C in her case) from the recorded temperature T. Then the log table is used to find log D.

Next Ann graphed log D as a function of t. The result is shown on Page 42. You can scale you axes similarly, although minor changes might be necessary on the vertical axis. As you can see, Ann's results do not quite lie on a straight line. Therefore the cooling was not quite exponential. Your results will probably have the same general appearance.

However, there is a linear function which approximates Ann's results well for part of the time interval. You can see that the points on the graph lie quite close to the line for the time interval from 3.5 minutes to 11 minutes. Try to find a linear section for your own data and draw the approximating line.

Now we use the technique of Section 8. First we find the equation of the straight line. The log D intercept is 1.67. The slope can be calculated from any two points on the line. Suppose we choose the points (0,1.67) and (12,1.12).

$$m \approx \frac{1.12 - 1.67}{12 - 0}$$

$$\approx -\frac{.55}{12}$$

$$\approx -.0458$$

Therefore the equation of the line is

$$\log D \approx -.0458t + 1.67$$

This means that

$$D \approx 10^{-.0458t + 1.67}$$

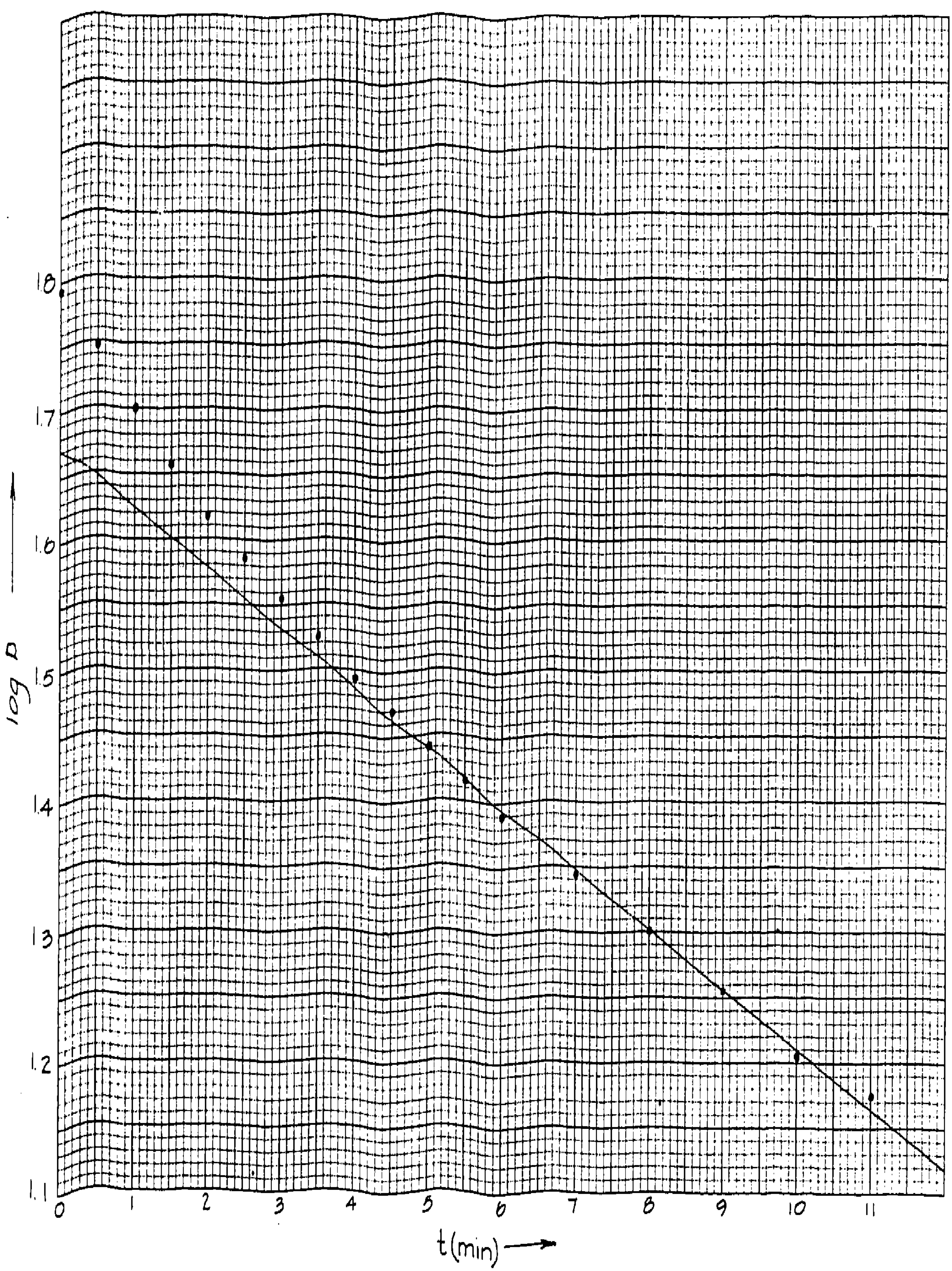
$$D \approx (10^{1.67})(10^{-.0458})^t$$

$$D \approx 47(.9)^t$$

This equation closely approximates Ann's results for the interval t = 3.5 to t = 11.

ROOM TEMPERATURE: 26° C

TIME (min)	TEMPERATURE (°C)	D	log D
0.0	88	62	1.792
0.5	82	56	1.748
1.0	76	50	1.699
1.5	71.5	45.5	1.658
2.0	67.5	41.5	1.618
2.5	64.5	38.5	1.585
3.0	62	36	1.556
3.5	59.5	33.5	1.525
4.0	57	31	1.491
4.5	55.5	29.5	1.470
5.0	53.5	27.5	1.439
5.5	52	26	1.415
6.0	50.5	24.5	1.389
7.0	48	22	1.342
8.0	46	20	1.301
9.0	44	18	1.255
10.0	42	16	1.204
11.0	41	15	1.176



1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
log D
0
1
2
3
4
5
6
7
8
9
10
11
t(min)

REVIEW PROBLEM SET 10:

Problems 1 - 3 refer to the function $y = 5(2^t)$.

1. Construct a table displaying the values of the function for values of t from 0 to 6.

2. Add a Δy column to the table and fill in the values.

3. For this function what is the value of k in the formula

$$\Delta y = ky \quad ?$$

4. (Multiple Choice) A given function satisfies the equation $\Delta y = ky$. The function is probably

a. linear b. quadratic c. exponential

Write a growth equation $\Delta y = ky$ for each of the following equations.

5. $y = 18(11)^t$ 6. $y = \frac{1}{5}(\frac{4}{3})^x$ 7. $y = 130(.99)^x$

8. An exponential equation has base 7 and scaling constant 40. Write the equation for the function (use variables y and t).

9. An exponential equation satisfies the following: the common ratio is $\frac{4}{5}$ and $y = 50$ when $t = 0$. Write the equation for the function.

10. An exponential satisfies the equation $\Delta y = 21y$ and the scaling constant is 37. Write the equation for the function.

11. A bacteria culture is increasing at the rate of 30% per hour. If the initial population was 10 cells, write an equation relating the population y to elapsed time t in hours.

12. Under the influence of an antibiotic, a bacteria population is decreasing by 20% per hour. If the initial population was 50 cells, write an equation relating the population y to elapsed time t in hours.

13. Find the exponential function corresponding to the table shown opposite.

t	y
0	$\frac{1}{3}$
1	$\frac{1}{2}$
2	$\frac{3}{4}$
3	$\frac{9}{8}$

14. A radioactive isotope has a half-life of 4 hours. If the initial amount is 64 grams, how much will remain at the end of 20 hours?

15. The isotope ${}_{15}P^{32}$ has a half-life of 14.3 days. How many days must pass for the amount of the isotope to be reduced to 30% of its original value?

16. A piece of charcoal is found to have an activity of 7 disintegrations per minute per gram of carbon. What is the approximate age of the charcoal?

The drug Meperidine has a volume of distribution of 50 liters and a half-life of 3 hours. Problems 17 - 20 are sequential. They must be done in order. Round answers to the nearest .1 $\frac{\text{mg}}{\text{liter}}$.

17. An initial dose of 60 mg is given. Compute the initial concentration is $\frac{mg}{liter}$.

18. What will the concentration be 4 hours after the initial dose?

19. Four hours after the initial dose another dose of 25 mg is given. What will the new blood concentration be?

20. What will the concentration be 4 hours after the second dose?

21. An epidemic in a city of 10^5 people is predicted to obey the equation

$$\frac{H}{I} = 10^5 \left(\frac{1}{10}\right)^t$$

where H = the number of healthy individuals, I = the number of infected individuals and t is time in days. Complete the table below. Round all the numbers in the right hand column so that there is only one non-zero digit.

t in days	$\frac{H}{I} = R$	I number of infected persons
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Convert each of the following equations to the form $y = Ar^t$ or $y = Ar^x$.

22. $y = 3^{3x}$

23. $y = (.1)^{2t}$

24. $y = 10^{.5t}$

25. $y = 4^{-2x}$

26. $\log y = 3t$

27. $\log y = -2t - 1$

28. $\log y = .6x + 2$

29. $\log y = -.1x + 2.7$

The following table shows the results of treating bacteria with an antibiotic solution.

t (time in hours)	0	.5	1.5	2.7	6	10
y (living bacteria)	430	410	350	330	240	170
log y						

30. Complete the log y row in the table.

31. a. Graph log y as a function of t and draw the best approximating line for the resulting points.

b. Find the exponential equation relating y and t.

SECTION 11: ANOTHER GRAPHING TECHNIQUE

11-1 Converting Another Type of Equation

At this point you have solved a number of problems requiring that you convert an equation such as

$$\log y = -3t + 2$$

into an exponential form. Now you will see how to convert another type of equation which arises often in applied problems. As an example, suppose the following equation is given

$$\log y = 2 \log t + 3$$

What will this equation become when the inverse function 10^x is applied to both sides? The steps are shown below.

$$10^{\log y} = 10^{2 \log t + 3}$$
$$y = (10^{2 \log t})(10^3)$$
$$y = 1000\ (10^{\log t})^2$$
$$y = 1000\ t^2$$

The answer is not an exponential function but rather a simple quadratic. However, the answer often is not so simple, as the following examples show.

EXAMPLE:

Convert the equation

$$\log y = -4 \log x + .29$$

SOLUTION:

$$10^{\log y} = 10^{-4 \log x + .29}$$
$$y = (10^{.29})(10^{-4 \log x})$$
$$y \approx 1.95\ (10^{\log x})^{-4}$$
$$y \approx 1.95\ x^{-4}$$

EXAMPLE:

Convert

$$\log y = -.43 \log t - 1.58$$

SOLUTION:

$$10^{\log y} = 10^{-.43 \log t - 1.58}$$
$$y \approx (10^{-1.58})(10^{-.43 \log t})$$
$$y \approx \left(\frac{1}{10^{1.58}}\right)(10^{\log t})^{-.43}$$
$$y \approx \frac{1}{38}\ t^{-.43}$$

11-2 Another Graphical Technique

Once the transformations of Section 11-1 are mastered it becomes possible to determine relations between two variables by plotting the logarithms of both variables.

EXAMPLE:

The following table shows the relationship between weight W in kilograms and body surface area S in square meters for several children. Find an equation for the relationship between the two variables.

W	2	3	4	5	6	7	8	9	10
S	.16	.21	.26	.30	.34	.38	.41	.45	.48

SOLUTION:

First we fill in rows for log W and log S.

log W	.301	.477	.602	.699	.778	.845	.903	.954	1.00
W	2	3	4	5	6	7	8	9	10
S	.16	.21	.26	.30	.34	.38	.41	.45	.48
log S	-.796	-.678	-.585	-.523	-.469	-.420	-.387	-.347	-.319

Next we plot log S as a function of log W and draw the best line through the resulting points. (The graph is shown on the following page.)

From the graph we find that the vertical intercept of the line is about -1 and the slope is about .68. This leads to the equation

$$\log S \approx .68 \log W + (-1)$$

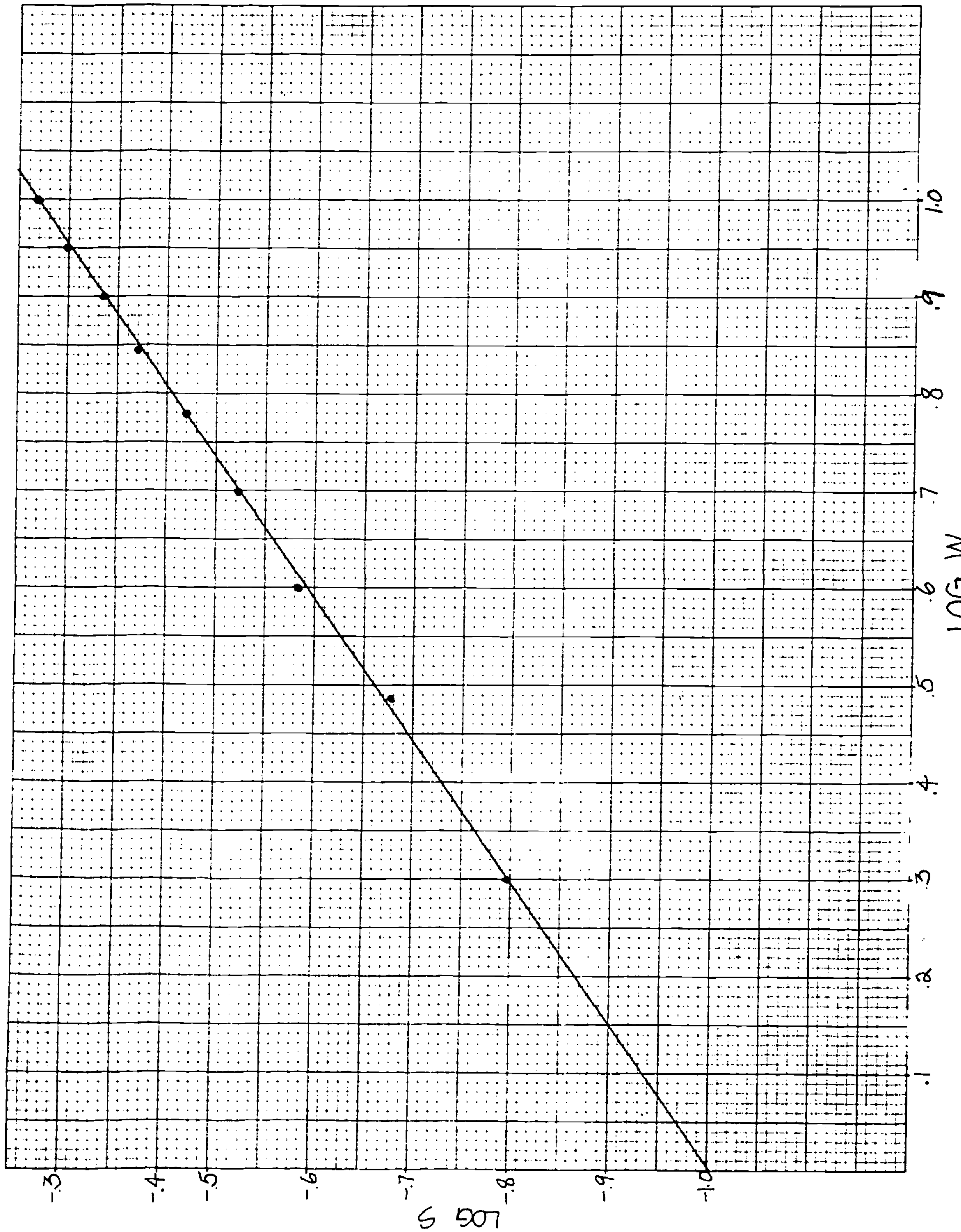
We now follow the same technique as in the examples to put the equation in exponential form.

$$S \approx (10^{\log W})^{.68}\ (10^{-1})$$

$$(.1)W^{.68}$$

11-3 Which Technique Should Be Used for a Given Set of Data?

How do we know which graphical method to use on a given table of data? That is, how do we know whether to find the log of one variable or the logs of both variables? In an experimental situation a mathematician might have to try both techniques (and several others) before he finds a satisfactory equation to describe the data. In the problem set it will always be clear which method to use.



PROBLEM SET 11:

Convert the following equations to the form $y = At^b$ or $y = Ax^b$. Convert all powers to their decimal equivalents.

1. $\log y = 2 \log t + 2$

2. $\log y = - \log t + 3$

3. log y = 5 log x - 2

4. log y = -7 log x - 3

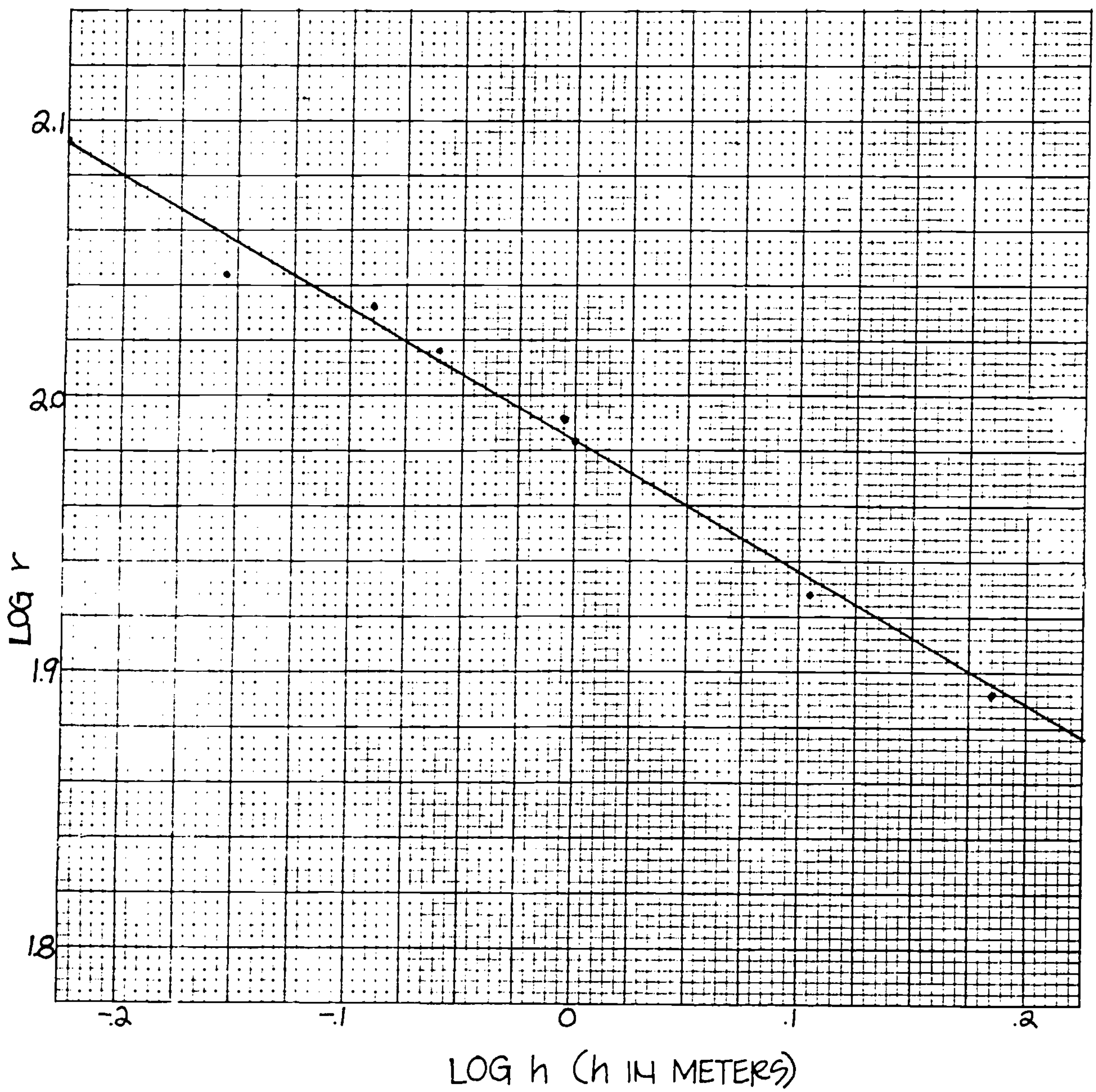
5. log y = 3.8 log x + 2

6. log y = -3 log x + .9

7. log y = 12.7 log t - .7

8. log y = -.32 log t + 1.47

The following graph shows the results of measuring pulse rate r and height h (in meters) for several individuals. The log of r has been plotted against the log of h and a best line drawn.



9. Find the equation of the line. (Note that the vertical axis is not at the left as usual, but rather in the middle.)

10. Find the corresponding exponential equation.

The following table shows the course of a chemical reaction involving a peptone.

t (hours)	.5	1	2	4	6
C (concentration)	47.5	35.5	28.0	21.5	18.0

11. Copy the table including a log t row and a log C row.

12. Graph log C as a function of log t and draw the best line through the resulting points.

13. Find the equation of your line.

14. Convert the equation.

The following table shows the relationship between sitting height S in meters and weight W in kg. for four individuals.

S	.87	.90	.92	.95
W	58	64	69	76.5

15. Copy the table including a log S row and a log W row.

16. Graph log W as a function of log S and draw the best line through the resulting points.

17. Find the equation of the line.

18. Convert the equation.

SECTION 12: METABOLISM OF ANIMALS

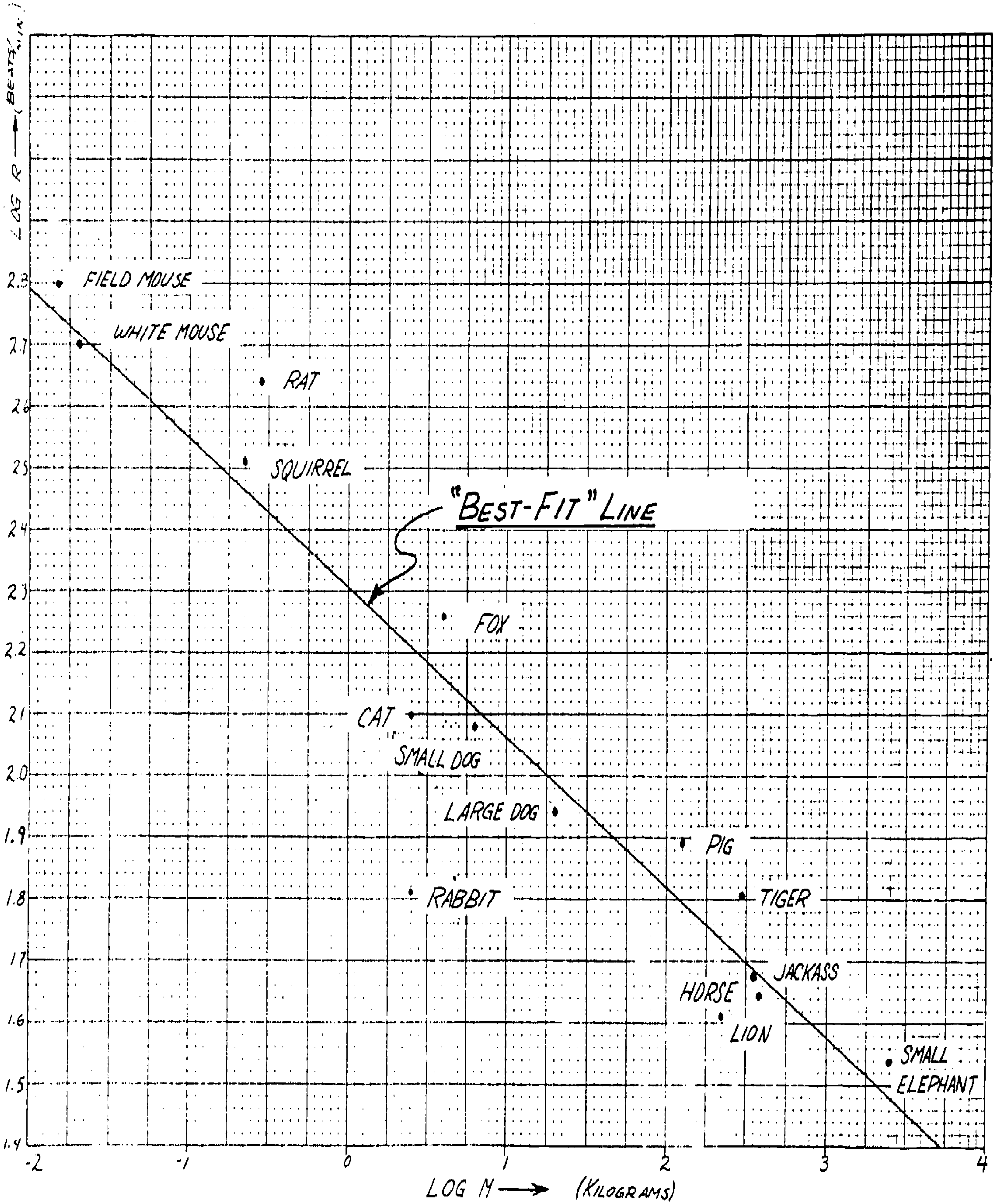
12-1 Heart Rate and Body Mass

Small animals live at a fast pace. They have fast pulse rates, and short lifespans. Also, they eat several times their weight in food every day and burn many more calories for each kilogram of body weight than larger animals do.

Larger animals live at a slower pace. They have slower heart rates and respiratory rates and longer lifespans and eat only a fraction of their weight in food each day.

These observations are hardly precise. It is no surprise that mathematicians have investigated such patterns with an eye to finding functional relationships. To start, let's concentrate on pulse rate. Figure 1 is a graph showing the relation between average body mass and average heart rate for various species of animals. We have plotted the common logarithm of body mass on the x-axis and the common logarithm of heart rate on the y-axis.

Field mice have an average mass of .015 kilogram and an average heart rate of 630 beats per minute. The logarithms are -1.82 and 2.80, so we plot the field mouse on the graph at the point (-1.82,2.80).


LOG R → (BEATS/MIN.)
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
FIELD MOUSE
WHITE MOUSE
RAT
SQUIRREL
"BEST-FIT" LINE
FOX
CAT
SMALL DOG
LARGE DOG
PIG
RABBIT
TIGER
JACKASS
HORSE
LION
SMALL ELEPHANT
-2
-1
0
1
2
3
4
LOG M → (KILOGRAMS)

A small elephant has a mass of 2500 kilograms and an average heart rate of 35 beats per minute. Taking logarithms, we get 3.40 and 1.54; so we plot the elephant at (3.40, 1.54). The elephant is farther to the right than the mouse because it is more massive. It is lower on the graph than the mouse because it has a lower heart rate.

The other species on the graph are plotted in the same way. The line which most closely fits all these points is the line

$$y \approx -.25x + 2.30$$

That is,

$$\log(\text{heart rate}) \approx -.25 \log(\text{mass}) + 2.30$$

12-2 Daily Heat Production and Body Mass

Heart rate isn't the only way to measure how fast an animal lives. We can also measure the amount of heat energy the animal produces in a day.

If we plot the average mass of various species against their average heat production rates, we find that many species are very close to the line

$$\log H \approx .75 \log M + 1.20$$

where H is the heat production rate in kilocalories per day and M is the mass in kilograms. (See Figure 2.)

This doesn't look much like the relation between heart rate and body mass--this line slopes upward, while the heart rate line slopes down. But suppose we are interested in how much heat a species produces per day for each kilogram of its body mass. Then we can get a relation which does look like the one between heart rate and mass.

An animal's daily rate of heat production per kilogram of mass is just $\frac{H}{M}$. And we know that

$$\log \left(\frac{H}{M}\right) = \log H - \log M$$

Combine this with

$$\log H \approx .75 \log M + 1.20$$

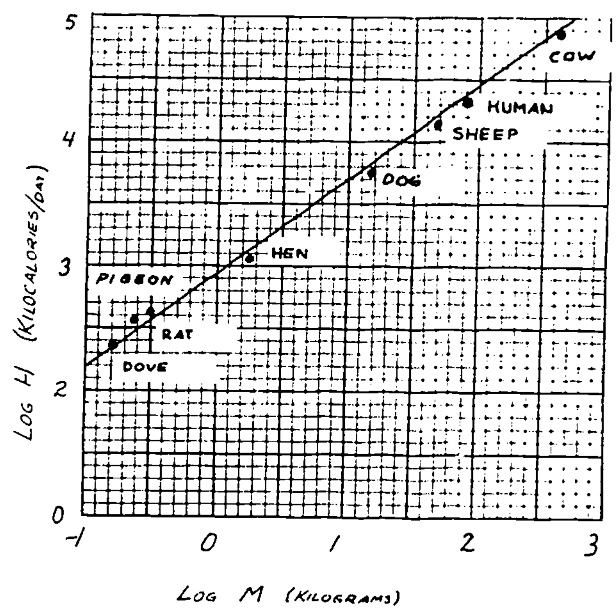


FIGURE 2

by substituting (.75 log M + 1.20) for (log H) in the first equation to get

$$\log \left(\frac{H}{M}\right) \approx (.75 \log M + 1.20) - \log M$$

$$\approx (.75 - 1) \log M + 1.20$$

$$\approx -.25 \log M + 1.20$$

So we see that an animal's daily heat production rate per kilogram of body mass is related to its mass in much the same way that its heart rate is related to its mass.

There is another property besides heart rate and heat production per kilogram which measures how fast an animal lives. It is respiration rate (number of breaths per minute). It turns out that respiration rate is also related to mass in much the same way as pulse rate.

12-3 An Explanation

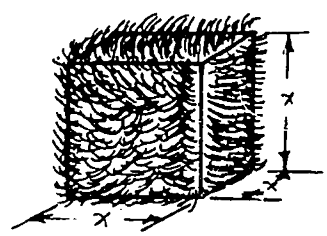
Why are all these properties related to body mass in this particular way? To answer this question, we first notice that heart rate, respiration rate and daily heat production per kilogram all depend on how quickly an animal loses body heat.

An animal that loses heat quickly has to produce heat quickly in order to keep its body temperature up. This means a high heat production per kilogram. It also means a high food intake (to supply the fuel) and a fast pulse and breathing rate (to stoke the furnaces). And it means a short lifespan (the machinery wears out quickly because it always runs at high speed).

An animal that loses heat slowly does not have to produce heat so quickly. It will have a lower daily heat production per kilogram, a slower pulse, and so forth.

Now it turns out that smaller animals lose body heat more quickly than larger animals. It's just a matter of geometry. An animal produces heat in its entire body mass, but it can only lose heat through its body surfaces. So the ratio of body surface area to body volume determines how much heat an animal produces and loses in a given length of time.

This ratio is always larger for small animals than for large ones. To see this, think of an animal as a fur-covered box of flesh (Figure 3). If the box measures x centimeters on each side, the volume of the box will be x^3 cubic centimeters, and the surface area will be $6x^2$ square centimeters (x^2 square centimeters each for the top, bottom and four sides).



surface area = $6x^2$ cm^2

volume = x^3cm^3

FIGURE 3: A cube-shaped animal

So the surface-to-volume ratio is

$$\frac{\text{surface area}}{\text{volume}} = \frac{6x^2}{x^3} = \frac{6}{x}$$

That is, the surface-to-volume ratio is inversely proportional to the linear dimensions (length, width, height) of the animal.

This is true even if the animal isn't cube-shaped. The surface-to-volume ratio will still be inversely proportional to the animal's size. The value of the proportionality constant will depend on the animal's shape.

No matter what shape an animal has, if we make the animal twice as big in each direction we will reduce its surface-to-volume ratio by half. If we make the animal half as big in each direction, the surface-to-volume ratio will double (see Figure 4).

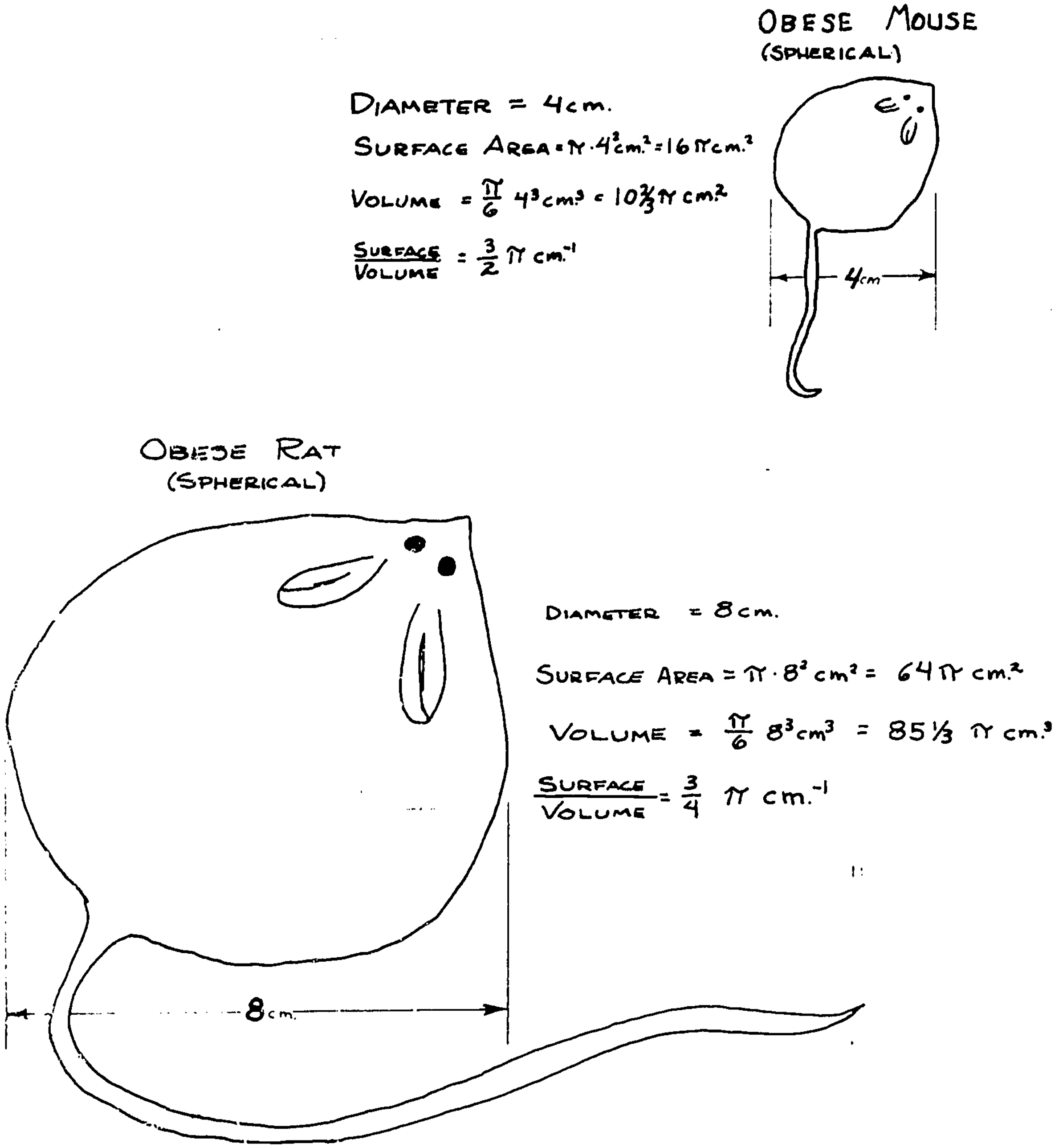


FIGURE 4

Now we make some assumptions. We assume that heart rate, heat production per kilogram and respiration rate are each proportional to heat loss. We also assume that heat loss rate is proportional to the surface-to-volume ratio. Then we have

$$R = \text{constant} \cdot \frac{1}{x}$$

where R is heart rate, daily heat production per kilogram or respiration rate. This is the same as

$$\log R = -\log x + \text{another constant}$$

We want to turn this equation into the equation

$$\log R \approx -.25 \log M + \text{some other constant}$$

In order to do this, we will have to find a relation between body size and body mass, that is, between x and M.

12-4 Body Mass and Heat Loss

It turns out that most animals have about the same density. That is, one cubic centimeter of animal weighs about the same, no matter what the animal. This means the animal's mass will be proportional to its volume.

But an animal's volume is proportional to the cube of its linear dimensions. That is, the linear dimensions (lenght, width, height) are proportional to the cube root of the mass. If we make the animal twice as big in each direction, it will have eight times the volume and eight times the mass. If we make it half as big in each direction, it will have only one-eighth the mass and volume.

This is easy to see for cube-shaped and sphere-shaped animals (Figure 3 and 4). But it is also true for animal-shaped animals.

Combining all our relations and assumptions, we get the following.

> The heart rate (or heat production per kilogram, respiratory rate and so forth) of an animal should be inversely proportional to the cube root of its mass.

In symbols, this is

$$R = \frac{K}{M^{\frac{1}{3}}}$$

where M is the mass, R is the heart rate (or heat production per kilogram, or whatever), and K is a proportionality constant. Another way to express this is

$$\begin{aligned}\log R &= \log \frac{K}{M^{\frac{1}{3}}}\\ &= \log K - \log M^{\frac{1}{3}}\\ &= \log K - \frac{1}{3}\log M\\ &\approx -.33 \log M + \text{constant}\end{aligned}$$

12-5 Back to the Real World

In the real world, it turns out that the relation is closer to

$$\log R \approx -.25 \log M + \text{constant}$$

This isn't exactly what we expected, but then we made a lot of assumptions to simplify the problem.

If you plot log R against log M for various species of animals, you will find that they don't all fall on this line but are scattered around it. This is not surprising. After all, some animals are more active than others of the same mass. The active animals have higher heart rates, respiratory rates, and so forth.

And some animals have better heat insulation than others of the same mass. They have thicker fur, or more fat. These animals lost less heat, and so should have lower heart rates, heat production and so on.

PROBLEM SET 12:

1. Look at the graph in Figure 1. The points on the "best fit" line correspond to species for which

$$\log R \approx -.25 \log M + 2.30$$

where R = average heart rate and M = average mass. Species on this line satisfy an exponential relation

$$R \approx AM^B$$

for some numbers A and B. Use the equation for the line and facts about logarithms to find the numbers A and B.

2. Elmo's hippo has a mass of 3500 kilograms. Assuming that hippos lie on the "best fit" line, what is the hippo's average heart rate to the nearest beat per minute? (Hint: Use the equation for the "best fit" line to find log R. Then find R.)

3. Elmo's average heart rate is 71 beats/minute. Assume that Elmo is on the "best fit" line. What is Elmo's mass (to the nearest tenth of a kilogram)?

4. Norbert's mass is 70 kilograms. Assume that Norbert is on the "best fit" line. What is his heart rate (to the nearest beat per minute)?

5. Assume that hippos lie on the line

$$\log H \approx .75 \log M + 1.20$$

a. How much heat does Elmo's hippo produce per day? Remember that the hippo's mass is 3500 kilograms. Compute your answer to the nearest 100 kilocalories per day.

b. Use the answer to Part a to compute the hippo's daily heat production per kilogram of body mass. (Compute to the nearest .1.)

Fat people are heavy for their height. Skinny people are light for their height. So a person's height and weight together determine both his shape and his size. This ought to be enough to determine his body surface area.

In fact, there is a formula which relates body surface area to body height and mass. This formula is

$$\log S \approx -0.69 + .40 \log M + .70 \log h$$

where S is the surface area in square meters, M is the body mass in kilograms, and h is the height in meters.

6. Elmo's height is about 1.7 meters and his mass is about 62 kg. Compute his body surface area, to the nearest .1 square meter.

7. Norbert's height is 2.0 meters and his mass is 70 kilograms. What is his body surface area, to the nearest .1 square meter?

8. Convert the formula for body surface area to an exponential formula of the form

$$S \approx E \cdot M^F \cdot h^G$$

for some numbers E, F and G.

SECTION 13: INFINITE SERIES

13-1 Examples of Infinite Sums

So far we have worked frequently with tables of logarithmic and trigonometric functions, but we have said nothing about how these tables are constructed. All such tables are computed by the use of infinite series. At this point we will discuss the subject of infinite series.

You are familiar with adding up a finite series of numbers. It may be a lot of work in certain cases, but we can be sure that there is a definite answer even if a thousand numbers are added. But what about adding an infinite series of numbers together? Is there ever any way of finding a definite answer in such a case?

In order to get a start on this question, suppose we look at the following two infinite series.

$$0 + 0 + 0 + 0 + \ldots$$
$$1 + 1 + 1 + 1 + \ldots$$

The first is the sum of an infinite number of zeros. The sum is zero. This is true because no matter how many zeros we add together, the answer is always zero.

The second series is a different matter. If we add the first two 1's, we get 2, the first three 1's give 3, the first one thousand 1's give 1000 and so on. The further we go, the larger the sum. The sky is the limit. Clearly this infinite series doesn't add up to any finite number. In summary, we can make sense out of the sums of certain infinite series, but others do not add up to any particular number.

As another example we find an infinite series that adds up to $\frac{1}{3}$. First we write the decimal expansion for $\frac{1}{3}$.

$$\frac{1}{3} = .33333\ldots$$

The decimal expansion is then written as an infinite sum as follows.

$$\frac{1}{3} = .3 + .03 + .003 + .0003 + .00003 + \ldots$$

13-2 A Word About Decimal Representations

In the above example we referred to the following equivalence.

$$\frac{1}{3} = .3333 \ldots$$

The three dots indicate that the pattern of 3's continues indefinitely.

If the above equation is multiplies by 2, another familiar equivalance is obtained.

$$\frac{2}{3} = .6666 \ldots$$

Suppose now that we multiply the first equation by 3.

$$\frac{3}{3} = .9999 \ldots$$

Since $\frac{3}{3} = 1$, you can see that there are two ways to write 1 in decimal form.

$$1.0000 \ldots$$
$$.9999 \ldots$$

In general, numbers with an infinite string of nines can be written in a simpler form. For example, the number

$$.5999\ \ldots$$

can be written more simply as .6.

13-3 Finding the Sum of an Infinite Series

We will now demonstrate a technique for finding the sum of an infinite series. As an example consider the following series. The sum of the series is denoted by S.

$$S = .9 + .09 + .009 + .0009 + \ldots$$

The behavior of this series can easily be studied by looking at the partial sums of the series. The first partial sum S_1 is the first term, the partial sum S_2 is the sum of the first two terms, and so on. The first three partial sums are shown below.

$$S_1 = .9 \qquad\qquad = .9$$

$$S_2 = .9 + .09 \qquad = .99$$

$$S_3 = .9 + .09 + .009 \qquad = .999$$

The pattern of the partial sums is clear. If 10 terms of the series are added the result will be a string of 10 nines, and so on. What if all the terms are added? The results must be an infinite string of nines. That is

$$S = .999\ \ldots$$

$$= 1$$

So the sum of the series is 1.

The following example is a little more complicated.

EXAMPLE:

Find the sum S, where

$$S = .45 + .045 + .0045 + \ldots$$

SOLUTION:

We start by writing down a few partial sums.

$$S_1 = .45 \qquad\qquad = .45$$

$$S_2 = .45 + .045 \qquad = .495$$

$$S_3 = .45 + .045 + .0045 \qquad = .4995$$

$$S_4 = .45 + .045 + .0045 + .00045 \qquad = .49995$$

Each additional term added results in another 9 in the sum. If we add all of the terms there will be an infinite number of nines following 4. Therefore

$$S = .4999\ \ldots$$
$$= .5$$

So the sum is .5 or $\frac{1}{2}$.

13-4 Series Which Don't Have Nice Sums

So far it has always been possible to find the exact sum of each series. The sums have turned out to be rational numbers like 1 or .5. However, many series don't have simple sums, and we cannot find the sum exactly but only to a certain number of decimal places. An example will give you an idea of how this works. Suppose we want to find the following sum.

$$S = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots$$

In order to tackle this problem, we construct a table. The left column contains the terms of the series and the middle column, the decimal equivalent of each term. The right hand column is a running total of the terms, or to put it another way, it is a list of successive partial sums.

TERM	DECIMAL EQUIVALENT	SUM
1	1.000000	1.000000
$\frac{1}{2^6}$	.015625	1.015625
$\frac{1}{3^6}$	.001372	1.016997
$\frac{1}{4^6}$	.000244	1.017241
$\frac{1}{5^6}$	.000064	1.017305
$\frac{1}{6^6}$	.000021	1.017326

Look at the column of decimal equivalents. The values are getting smaller. All the rest of the terms of the series will have decimal equivalents with at least 4 zeros after the decimal point.

Now look at the column of partial sums. The sums are growing, but not very rapidly. The last 3 partial sums all agree to 3 decimal places. We can make an intelligent guess that

$$S \approx 1.017$$

to 3 decimal places.

What about the exact sum of the series? Can you guess what it should be? If so, you are a genius, because this question puzzled the very best mathematicians for several decades. The answer was finally published in 1748 by the great mathematician Euler (pronounced "oiler"). The answer is

$$\frac{\pi^6}{945} \approx 1.017326316$$

TABLE 13

Decimal Equivalents for Some Fractions
(to be used in Problem Set 13)

$\frac{1}{3} = .333\ldots$	$\frac{1}{8} = .125$	$\frac{1}{9} = .111\ldots$	$\frac{5}{9} = .555\ldots$
$\frac{2}{3} = .666\ldots$	$\frac{3}{8} = .375$	$\frac{2}{9} = .222\ldots$	$\frac{7}{9} = .777\ldots$
$\frac{1}{6} = .1666\ldots$	$\frac{5}{8} = .625$	$\frac{4}{9} = .444\ldots$	$\frac{8}{9} = .888\ldots$
$\frac{5}{6} = .8333\ldots$	$\frac{7}{8} = .875$		
$\frac{1}{11} = .090909\ldots$	$\frac{4}{11} = .363636\ldots$	$\frac{7}{11} = .636363\ldots$	$\frac{10}{11} = .909090\ldots$
$\frac{2}{11} = .181818\ldots$	$\frac{5}{11} = .454545\ldots$	$\frac{8}{11} = .727272\ldots$	
$\frac{3}{11} = .272727\ldots$	$\frac{6}{11} = .545454\ldots$	$\frac{9}{11} = .818181\ldots$	

PROBLEM SET 13:

In solving t‵ : ‵ollowing problems you will often need to refer to TABLE 13 above. It will help ỵ. ‵. decide what numbers are represented by certain decimal expressions.

1. a. Write down the first <u>four</u> partial sums for the series

$$S = .6 + .06 + .006 + \ldots$$

   b. What must the sum S be? Use TABLE 13 to express S as a fraction.

2. a. Write down the first <u>three</u> partial sums for the series

$$S = .27 + .0027 + .000027 + \ldots$$

   b. What must the sum be?

For Problems 3 through 5, refer to the following series.

$$S = .18 + .018 + .0018 + .00018 + \ldots$$

3. Write down the first <u>four</u> partial sums.

4. As each new term is added, the effect is to add another ____(?)____ to the decimal expansion.

5. What must the sum S be? Leave it as a decimal.

In Problems 6 through 11 write the sum S as a fraction or mixed number. You will need to refer often to TABLE 13.

6. $S = .36 + .036 + .0036 + \ldots$

7. $S = .75 + .075 + .0075 + \ldots$

8. $S = 2.2 + .22 + .022 + \ldots$

9. $S = 3.2 + .32 + .032 + \ldots$

10. $S = 11.25 + .1125 + .001125 + \ldots$

11. $S = 2.34 + .0234 + .000234 + \ldots$

12. Suppose we want to find the following sum.

$$S = 1 - \frac{1}{5^3} + \frac{1}{10^3} - \frac{1}{15^3} + \frac{1}{20^3} - \cdots$$

Notice that there are negative terms in this sum. The table below shows the decimal equivalents of the first few terms.

TERM	APPROXIMATE DECIMAL EQUIVALENT	SUM
1	1.00000	
$-\frac{1}{5^3}$	- .00800	
$\frac{1}{10^3}$	.00100	
$-\frac{1}{15^3}$	- .00030	
$\frac{1}{20^3}$	.00012	
$-\frac{1}{25^3}$	- .00006	

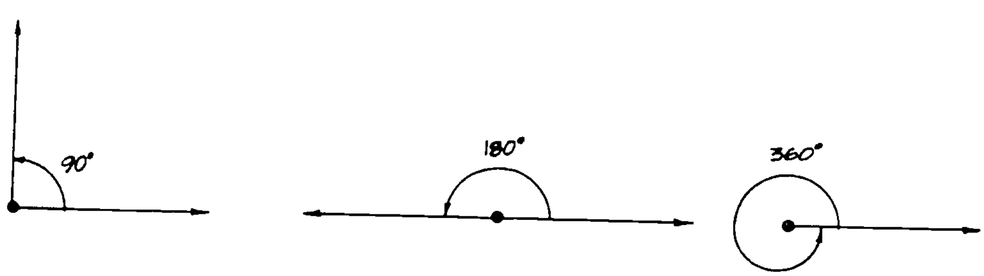
a. Find the numbers that go in the sum column.

b. Based on the partial sums, what should S be, rounded to three decimal places?

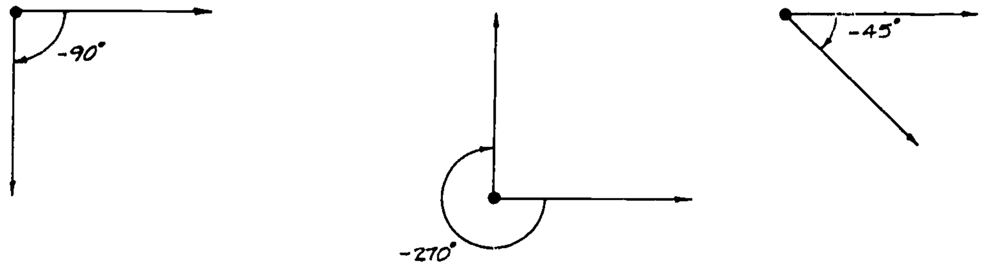
SECTION 14: THE SINE AND COSINE FUNCTIONS

14-1 Review of Trigonometric Functions

In Unit IV of Biomedical Mathematics you studied the trigonometric funcitons sine and cosine. In this section you will see how these functions are related to infinite series. Let us recall how the sine and cosine were defined. Both are functions of angles. Degrees are the most familiar measure of angles. A right angle has 90°, a straight angle, 180°, and a full circle, 360°.



Notice that these angles all show a counterclockwise rotation. Angles which arise from a clockwise rotation are considered to be negative.



Another important angle measure is the radian. 180° is equivalent to π radians. Therefore 360° is equivalent to 2π radians and so on. The following relationships can be used to convert degrees to radians and vice versa.

$$\text{angle in radians} = \frac{\pi \text{ radians}}{180 \text{ degrees}} \times (\text{angle in degrees})$$

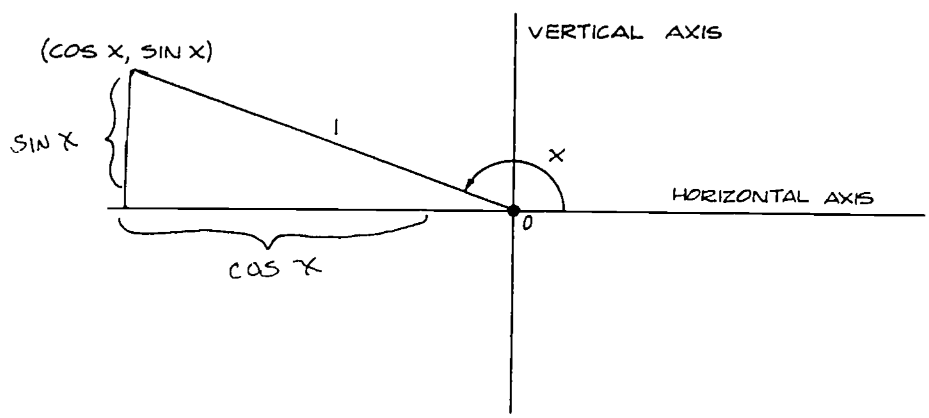
$$\text{angle in degrees} = \frac{180 \text{ degrees}}{\pi \text{ radians}} \times (\text{angle in radians})$$

For example, how many degrees are there in 1 radian? In order to find out, the second formula can be used.

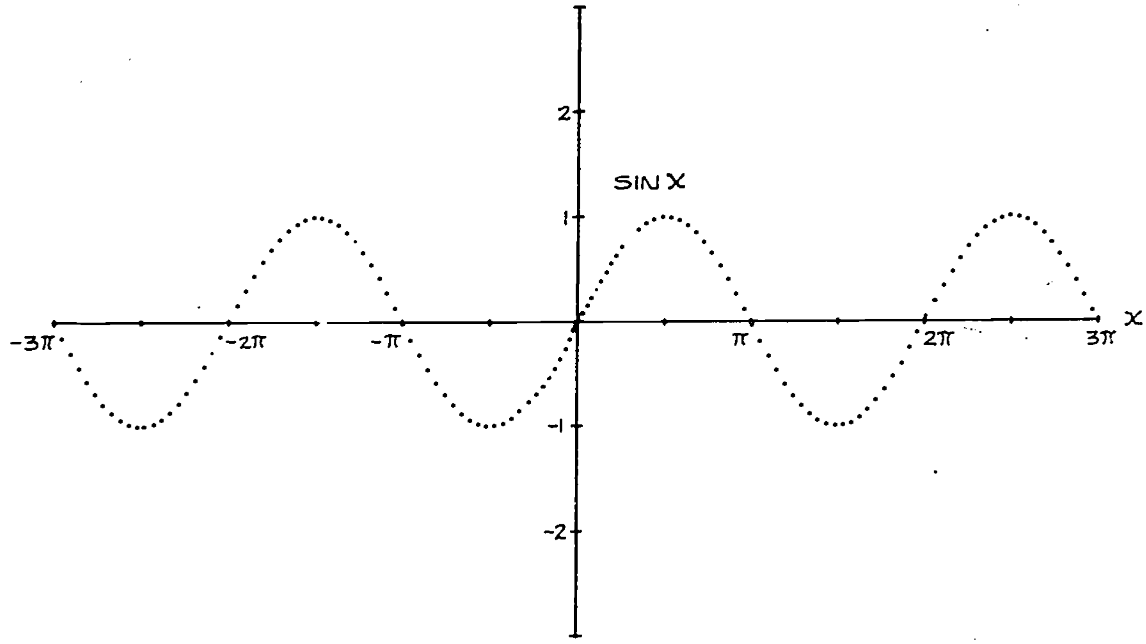
$$\text{angle in degrees} = \frac{180 \text{ degrees}}{\pi \text{ radians}} \times (1 \text{ radian})$$

$$\approx 57.3°$$

Recall how the trigonometric functions sine and cosine are defined. Suppose we want the values of sine and cosine for an angle x. We begin on the positive side of the horizontal axis and construct the angle x. Then we draw a vector of length 1 in the direction x. The horizontal coordinate of the vector's tip is cos x and the vertical coordinate is sin x.



The graph of the sine function is shown on the following page. You worked with graphs like this in the unit on sound. Notice that the angles are expressed in radians.



As you can see, the pattern of the sine curve for negative angles is just a continuation of the pattern for positive angles. This pattern continues indefinitely in both the positive and negative directions.

14-2 Polynomial Functions and the Sine Function

In the case of certain functions it is quite easy to find y when x is given. This is true of a linear function such as

$$y = 3x + 2$$

or of a quadratic function such as the following.

$$y = 8x^2 - \frac{1}{3}x + \frac{1}{120}$$

When x is given you need only substitute, find the indicated powers and it all comes down to a multiplication and addition problem.

These last two functions are examples of polynomial functions with integral exponents, that is, functions which are formed by adding integral powers of x, each multiplied by some real number. The largest power of x appearing is called the degree of the polynomial. For example,

$$y = 1 - 8x^2 + 29x^{10} - 76x^{122}$$

is a polynomial function of degree 122.

The functions sine and cosine present a constrast to polynomial functions. They are hard to evaluate. If you were asked to figure out sin 86° you would probably be hard pressed to do it. You are accustomed to referring to a table like Table 14 (located at the end of this section). But how are the values in the table computed?

In order to answer this question we will indulge in a little wishful thinking. If only the sine function were a polynomial it would be easy to evaluate. Perhaps the sine function is in fact a polynomial and we just don't know it.

Could the sine function be a linear function? No, because the graph of the sine is not a straight line. Since the graph of the sine is not a parabola, the sine cannot be a quadratic function either. Continuing along these lines, mathematicians have shown that no matter how large an integer n we choose, the sine function cannot be a polynomial of degree n.

In the end, however, this line of attack works. The sine is not a polynomial of <u>finite</u> degree, but it is a polynomial of <u>infinite</u> degree.

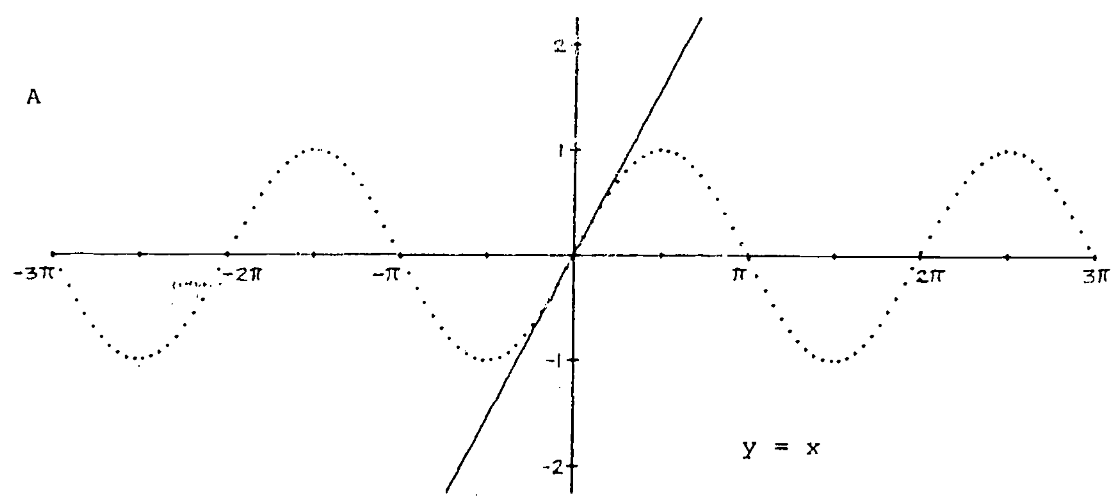
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

As you can see, only the odd powers of x appear. The signs of the terms alternate between positive and negative. The expressions in the denominations are factorials like the ones you worked with in the study of permutations and combinations. It is important to note at the start that the equation is true only if <u>x is expressed in radian measure</u>.

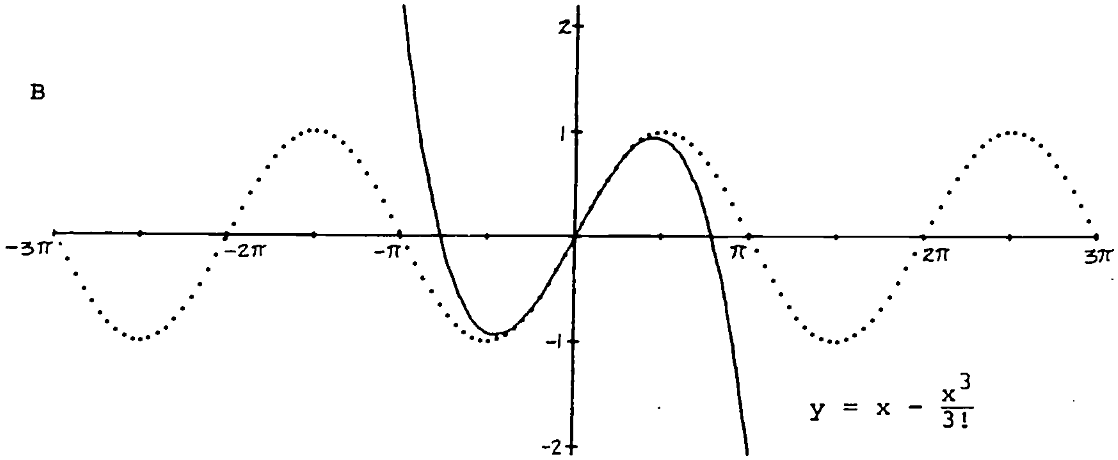
By now you have worked with infinite sums, but they were always sums of numbers rather than powers of x. In this case, the partial sums will be the polynomials x, $x - \frac{x^3}{3!}$ and so on. In order to see how the partial sum polynomials approximate sin x, the following graphs are included. Each graph shows sin x and the polynomial together. The graphs were drawn by a computer connected with a plotting device.

As more terms are added, each polynomial follows the sine graph a little farther before it "takes off." For example, on Graph D, the polynomial coincides almost exactly with sin x in the interval $-\pi$ to π. On the last graph, Graph K, the polynomial and sin x agree closely across almost all of the domain shown. You should keep in mind that even where the graphs appear to coincide, it is not usually because the functions are identical. Instead it is because the difference between the functions is too small to show up on the graph.

A

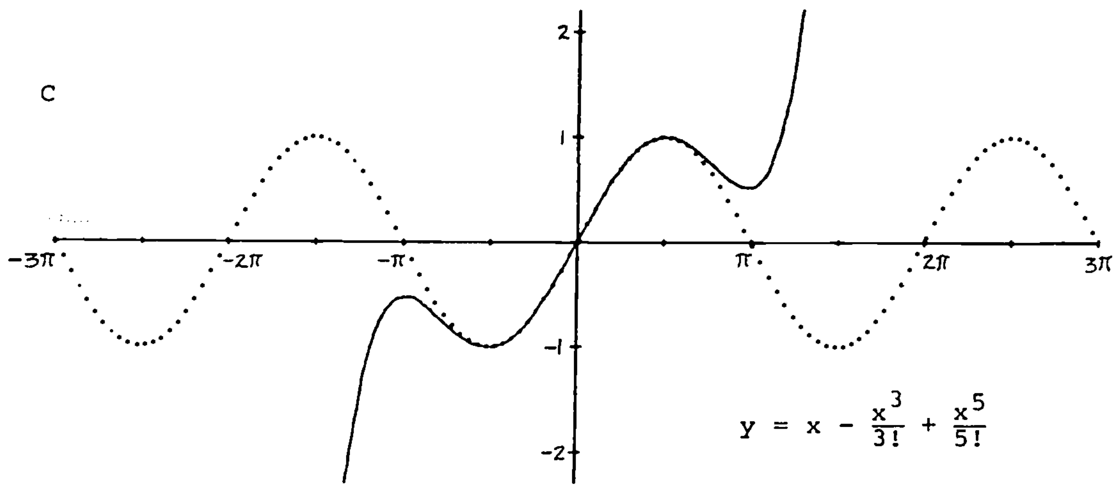


B



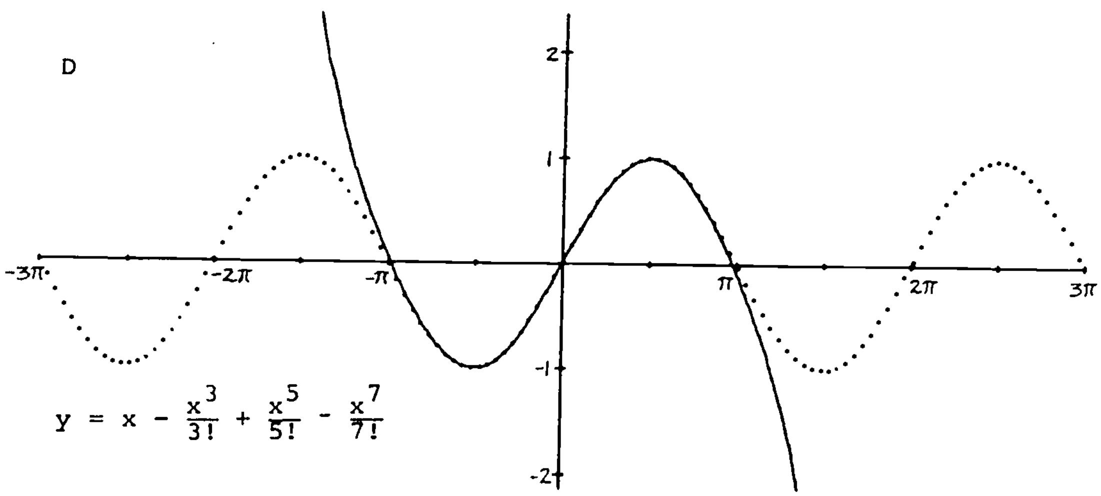
$$y = x - \frac{x^3}{3!}$$

C



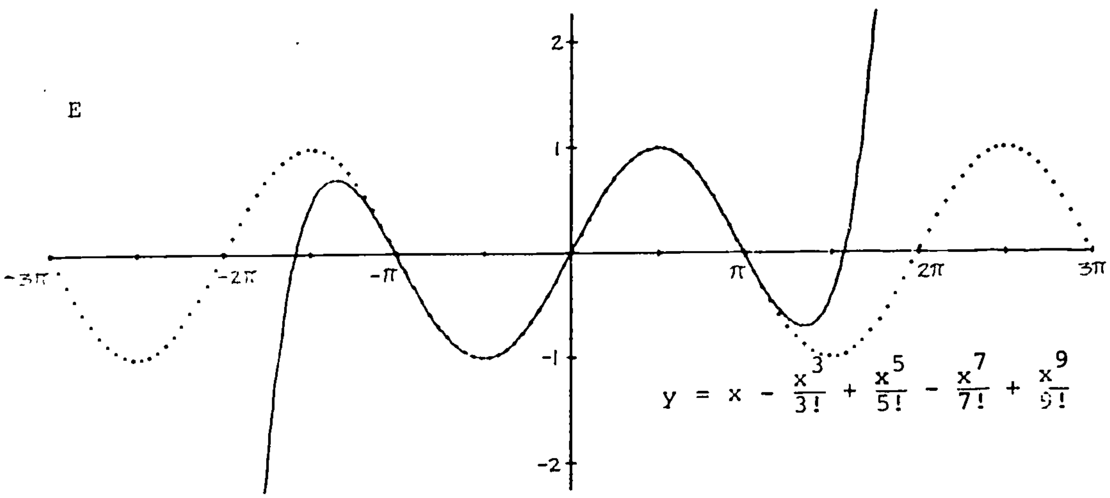
$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

D



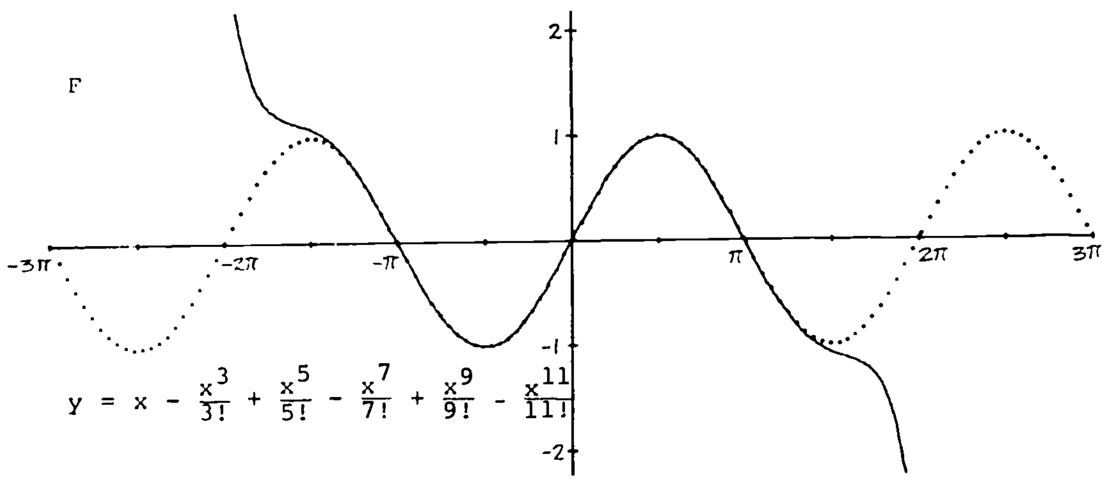
$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

E



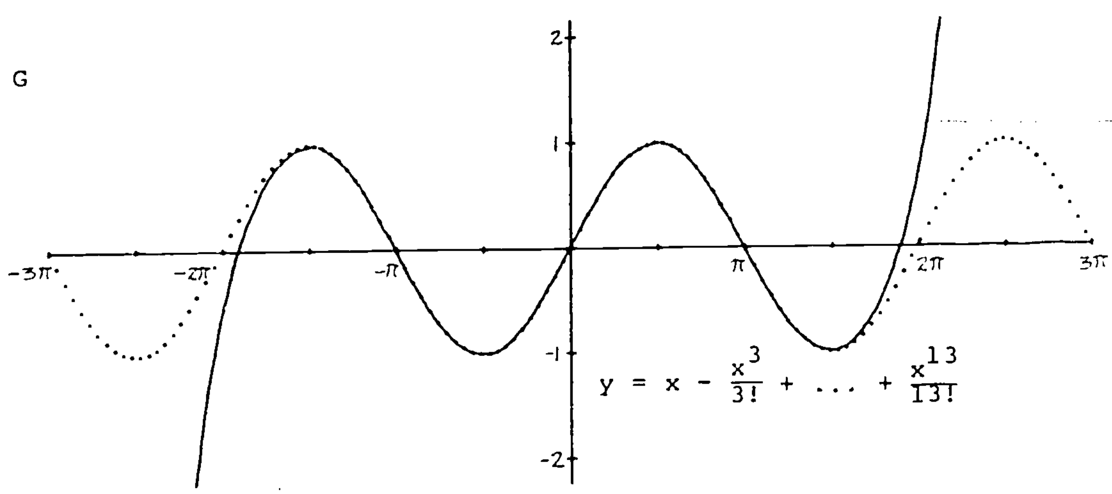
2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

F



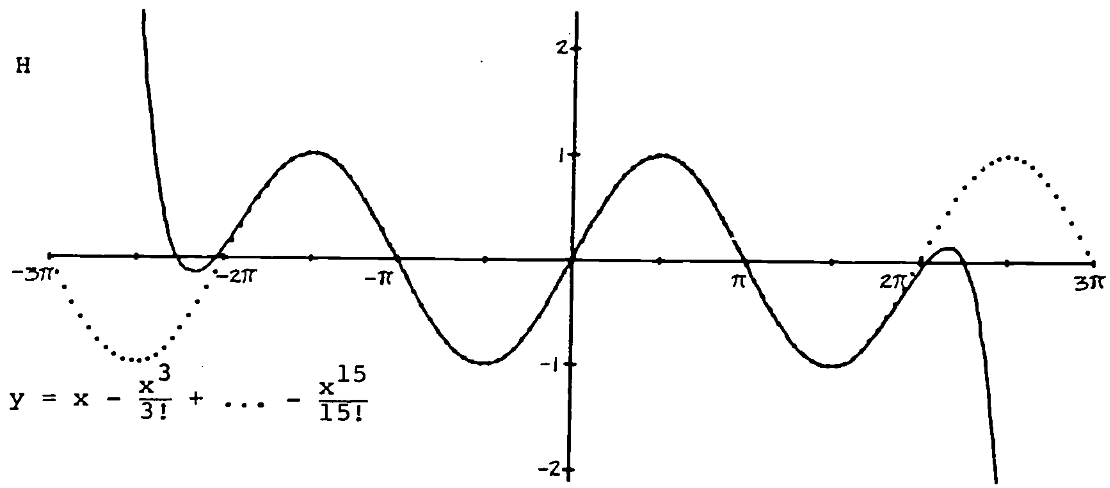
2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

G

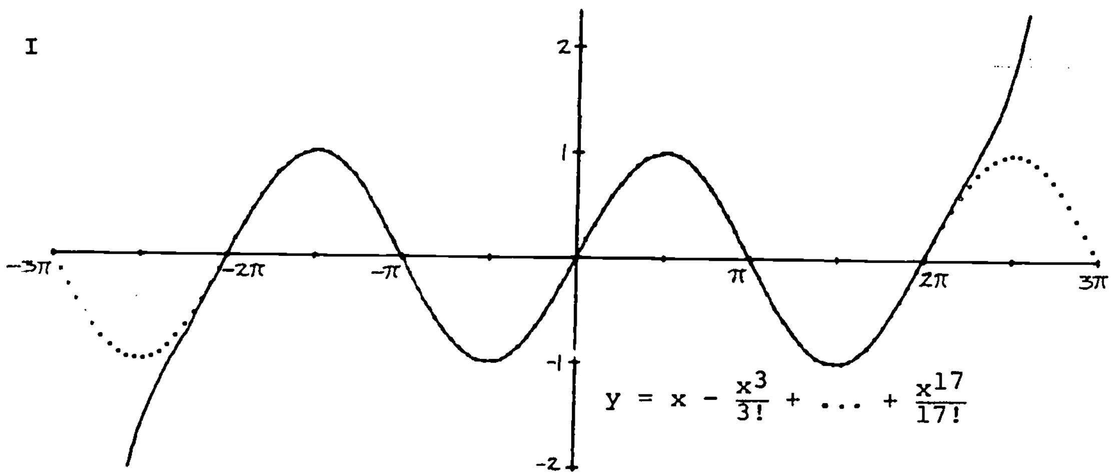


2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \ldots + \frac{x^{13}}{13!}$

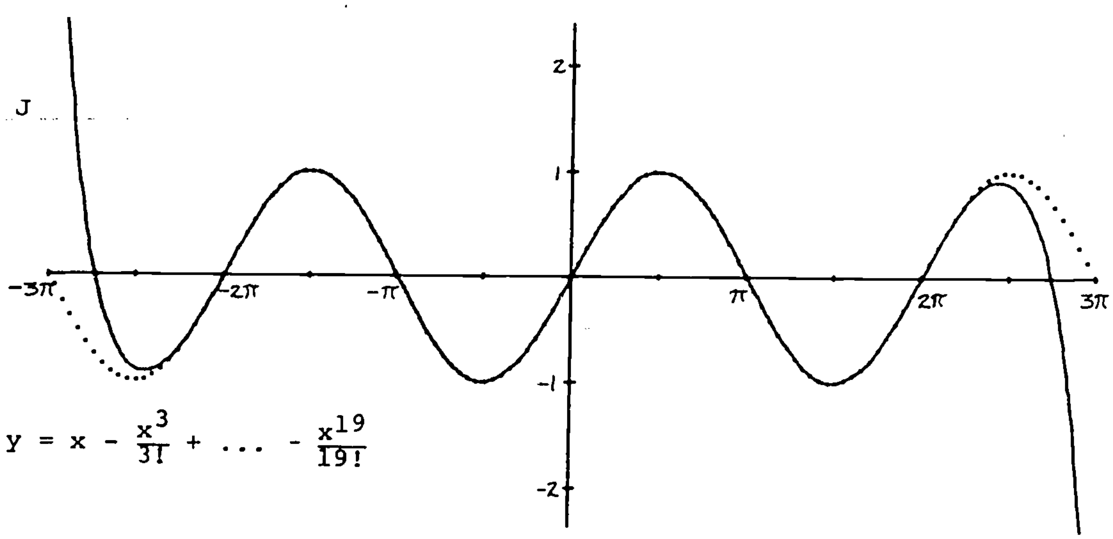
H

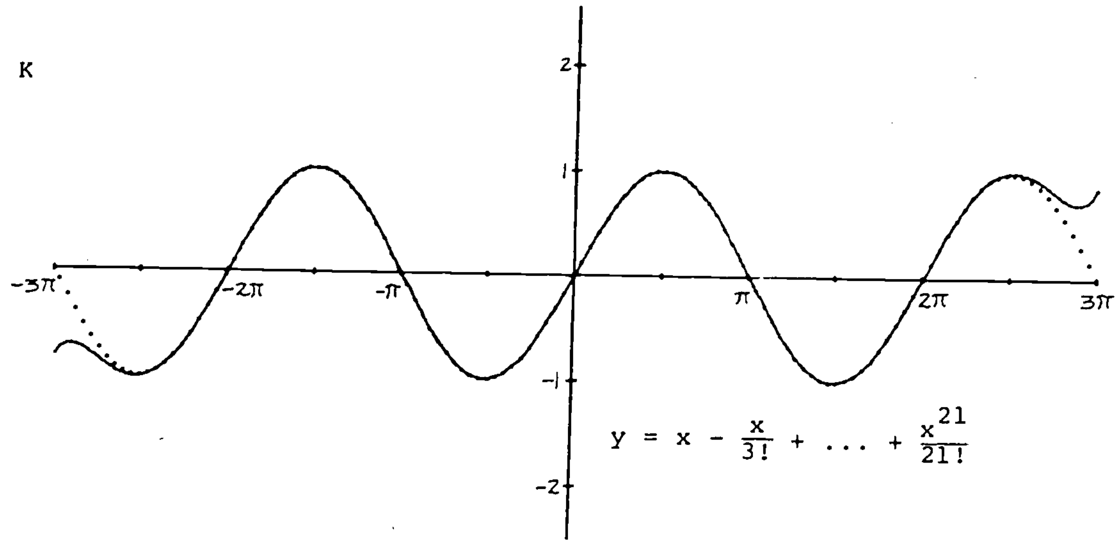

2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \dots - \frac{x^{15}}{15!}$

I


2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \dots + \frac{x^{17}}{17!}$

J


2
1
-1
-2
-3π
-2π
$-\pi$
π
2π
3π
$y = x - \frac{x^3}{3!} + \dots - \frac{x^{19}}{19!}$



PROBLEM SET 14:

Refer to Graphs A through K in the following problems.

1. The graph of the third partial sum (see Graph C) agrees closely with sin x for x in the interval

a. $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ b. $-\pi$ to π c. -2π to 2π

2. The seventh partial sum (Graph G) coincides closely with sin x for x in the interval (largest interval that applies)

a. $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ b. $-\pi$ to π c. $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$ d. -2π to 2π

3. Which is the first partial sum that appears to coincide with sin x over the entire interval -2π to 2π?

4. The eleventh partial sum (Graph K) appears to coincide with sin x over the interval ____(?)____.

5. Look at Graph B. You can see that $\sin x \approx x - \frac{x^3}{3!}$ for x between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. This fact can be used to get an approximate value for the sine of .611 radian.

a. Compute $x - \frac{x^3}{6}$ when x = .611. $[(.611)^3 \approx .228]$

b. What value appears in Table 14 for the sine of .611 radian?

c. What is the difference between the answers to Parts a and b?

d. .611 radians corresponds to an angle of ____(?)____ degrees.

6. The infinite series for the sine can easily be used to find the sine of 1 radian. We simply substitute 1 for x.

$$\sin 1 = 1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \frac{1^7}{7!} + \ldots$$
$$= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots$$

a. Use the series to compute sin 1. Express your answer to three decimal places. The table shown opposite will be useful.

b. Estimate the sine of 1 radian using Table 14.

c. What is the difference between the answers in Parts a and b?

N	$\frac{1}{N!}$
1	1.0000
2	.5000
3	.1667
4	.0417
5	.0083
6	.0014
7	.0002

7. There is also an infinite polynomial for the cosine function.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

a. Use the series to compute the cosine of 1 radian. (The table of Problem 6 contains the values you need.) Express your answer to three decimal places.

b. Use Table 14 to estimate the cosine of 1 radian.

c. What is the difference between the answers to Parts a and b?

You have now seen that the sine and cosine functions are infinite polynomials. It is easy to find the derivative of a polynomial. This fact can be used to find the derivatives of the sine and cosine. The following problems show how this is done.

8. Show that $\frac{3x^2}{3!} = \frac{x^2}{2!}$

9. Show that $\frac{5x^4}{5!} = \frac{x^4}{4!}$

10. The infinite series for sine and cosine are as follows.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ $\qquad$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Show that $\frac{d(\sin x)}{dx} = \cos x$ by finding the derivative of the polynomial for sin x.

11. Show that $\frac{d(\cos x)}{dx} = -\sin x$.

12. In the unit on sound you worked with functions of the form y = sin kx. It is easy to find an infinite polynomial for sin kx. We just substitute kx for x in the polynomial for sin x.

$$\sin kx = kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + \ldots$$
$$= kx - \frac{k^3x^3}{3!} + \frac{k^5x^5}{5!} - \frac{k^7x^7}{7!} + \ldots$$

Find the infinite polynomial for cos kx.

13. Take the derivative of the sin kx polynomial and use the distributive law to show that

$$\frac{d(\sin kx)}{dx} = k \cos kx$$

TABLE OF TRIGONOMETRIC FUNCTIONS

TABLE 14

Degrees	Radians	Cosine	Sine	Tangent	Degrees	Radians	Cosine	Sine	Tangent
0	.000	1.000	0.000	0.000					
1	.017	1.000	.018	.018	4	.803	.695	.719	1.036
2	.035	0.999	.035	.035	47	.820	.682	.731	1.072
3	.052	.999	.052	.052	48	.838	.669	.743	1.111
4	.070	.998	.070	.070	49	.855	.656	.755	1.150
5	.087	.996	.087	.087	50	.873	.643	.766	1.192
6	.105	.995	.105	.105	51	.890	.629	.777	1.235
7	.122	.993	.122	.123	52	.908	.616	.788	1.280
8	.140	.990	.139	.141	53	.925	.602	.799	1.327
9	.157	.988	.156	.158	54	.942	.588	.809	1.376
10	.175	.985	.174	.176	55	.960	.574	.819	1.428
11	.192	.982	.191	.194	56	.977	.559	.829	1.483
12	.209	.978	.208	.213	57	.995	.545	.839	1.540
13	.227	.974	.225	.231	58	1.012	.530	.848	1.600
14	.244	.970	.242	.249	59	1.030	.515	.857	1.664
15	.262	.966	.259	.268	60	1.047	.500	.866	1.732
16	.279	.961	.276	.287	61	1.065	.485	.875	1.804
17	.297	.956	.292	.306	62	1.082	.470	.883	1.881
18	.314	.951	.309	.325	63	1.100	.454	.891	1.963
19	.332	.946	.326	.344	64	1.117	.438	.899	2.050
20	.349	.940	.342	.364	65	1.134	.423	.906	2.145
21	.367	.934	.358	.384	66	1.152	.407	.914	2.246
22	.384	.927	.375	.404	67	1.169	.391	.921	2.356
23	.401	.921	.391	.425	68	1.187	.375	.927	2.475
24	.419	.914	.407	.445	69	1.204	.358	.934	2.605
25	.436	.906	.423	.466	70	1.222	.342	.940	2.747
26	.454	.899	.438	.488	71	1.239	.326	.946	2.904
27	.471	.891	.454	.510	72	1.257	.309	.951	3.078
28	.489	.883	.470	.532	73	1.274	.292	.956	3.271
29	.506	.875	.485	.554	74	1.292	.276	.961	3.487
30	.524	.866	.500	.577	75	1.309	.259	.966	3.732
31	.541	.856	.515	.601	76	1.326	.242	.970	4.011
32	.559	.848	.530	.625	77	1.344	.225	.974	4.331
33	.576	.839	.545	.649	78	1.361	.208	.978	4.705
34	.593	.829	.559	.675	79	[illegible]	.191	.982	5.145
35	.611	.819	.574	.700	80	[illegible]	.174	.985	5.671
36	.628	.809	.588	.727	81	1.414	.156	.988	6.314
37	.646	.799	.602	.754	82	1.431	.139	.990	7.115
38	.663	.788	.616	.781	83	1.449	.122	.993	8.144
39	.681	.777	.629	.810	84	1.466	.105	.995	9.514
40	.698	.766	.643	.839	85	1.484	.087	.996	11.43
41	.716	.755	.656	.869	86	1.501	.070	.998	14.30
42	.733	.743	.669	.900	87	1.518	.052	.999	19.08
43	.751	.731	.682	.933	88	1.536	.035	.999	28.64
44	.768	.719	.695	.966	89	1.553	.018	1.000	57.29
45	.785	.707	.707	1.000	90	1.571	.000	1.000	undefined

SECTION 15: EULER'S NUMBER

15-1 Infinite Series for Exponential Functions

In the last section we pointed out that tables of trigonometric functions are calculated by means of infinite degree polynomials, called infinite series. The specific series for sine and cosine were discussed. In this section we will explore the infinite series that are used to compute the values of exponential functions. Let us begin with the following three series, which can be obtained through methods of calculus.

$$5^x \approx 1 + 1.609x + \frac{(1.609)^2x^2}{2!} + \frac{(1.609)^3x^3}{3!} + \frac{(1.609)^4x^4}{4!} + \ldots$$

$$3^x \approx 1 + 1.099x + \frac{(1.099)^2x^2}{2!} + \frac{(1.099)^3x^3}{3!} + \frac{(1.099)^4x^4}{4!} + \ldots$$

$$2^x \approx 1 + .693x + \frac{(.693)^2x^2}{2!} + \frac{(.693)^3x^3}{3!} + \frac{(.693)^4x^4}{4!} + \ldots$$

In some respects, these series resemble those for the sine and cosine. The familiar factorials 1!, 2!, etc. appear in the denominators. However these series are not as simple because of the messy constants in each term.

Fortunately the constants provide a clue to a simpler series. Notice that the constants increase as the base gets larger and that the constant for 2^x is less than one while the constant for 3^x is greater than one. Perhaps there is a number e between 2 and 3 such that the constant for e^x is exactly one. In fact, there is such a number, and we can write.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

This series is much simpler than the others. Therefore it is much easier to work with.

15-2 e for Euler

Before we determine the numerical value of the number e, a few words about its origin are in order. The number e is called Euler's (pronounced "oilers") number. This may be the first time that you have ever heard of Euler's number, but if you decide to continue your mathematical education beyond high school you will hear much more about Euler himself. He was a giant in the history of the development of mathematics.

Euler came on the scene early in the development of calculus. He used the then new tools of the calculus to greatly expand mathematical knowledge. He was by far the most productive of all mathematicians. A group in his native Switzerland is putting together a complete set of his works. It is not finished yet, but they figure that it will run to 75 volumes of about 600 pages each, or about 45,000 pages. This works out to a production for Euler of roughly 3 printed pages of mathematics per day over a span of roughly 50 years! Furthermore, it was all hard, all brilliant and almost without error. In the case of Euler a high rate of production did not imply a sacrifice in quality.

Surely this man must have been some freakish weirdo. One suspects that Euler would have had some of the personality quirks that today people associate with mad scientists. For example, an introverted personality and a demented laugh. Not so. Euler was very much a family man. During much of his productive life he supported a household that numbered 13. This number included parents, children, grandchildren, etc. He was reportedly able to develop new mathematics with one child in his lap and another one tugging on his arm.

In his later years Euler became blind. This would seem to spell the end of a mathematician's productive life, but not so for Euler. He learned to dictate mathematics. His rate of production barely faltered. In fact, he continued to develop new mathematics for 12 years while completely blind. Only death was able to stop Euler's production of mathematics. He died in 1783.

15-3 The Value of e

In order to compute the value of e, we substitute $x = 1$ in the series for e^x. Since $e^1 = e$, this will give us the valu of e.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \ldots$$

It is impossible to evaluate e exactly by calculating a finite number of terms. However, it is always possible to calculate e as exactly as needed for any situation. This is because $\frac{1}{n!}$ gets very small very quickly as n increases. For example,

$$\frac{1}{60!} \approx 1.2 \times 10^{-82}$$

The first ten digits of e are

$$e \approx 2.718281828$$

This may look like a repeating decimal, but it isn't. The ...1828... pattern stops after the last 8. As we have pointed out, in the series for e^x there are no clumsy, hard to remember, constants hanging around. However, the base e is not as "nice" as we might have hoped for. It is much more difficult to square 2.718281828... than it is to square 2, for example. Just as in most other things, we can't always get everything we want. In this case, we cannot have both an aesthetically pleasing series and a nice, easy to work with base.

15-4 Does the Series for e^x Have the Property $(e^x)(e^v) = e^{x+v}$?

The question that we wish to deal with at this point is how an infinite polynomial can behave like an exponential function. Remember that the product of two powers of r is simply r raised to the sum of powers. In general

$$r^x \cdot r^v = r^{x+v}$$

The polynomial series for e^x should also have this property. Specifically,

$$e^x \cdot e^v = e^{x+v}$$

In terms of the series, the question is, does

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)(1 + v + \frac{v^2}{2} + \ldots) =$$

$$1 + (x + v) + \frac{(x + v)^2}{2!} + \frac{(x + v)^3}{3!} + \ldots ?$$

The truth of this equation is not immediately obvious. The demonstration that it is true requires only a knowledge of the distributive law. Recall that

$$a(b + c) = ab + ac$$

We can apply the distributive law to the problem of multiplying two infinite polynomials as follows. We start out by ignoring the fact that e^x is an infinite polynomial. This simple-minded approach leads to the following development.

$$(e^x)(e^v) = e^x \; (1 + v + \frac{v^2}{2!} + \ldots)$$

$$= e^x(1) + e^x(v) + e^x(\frac{v^2}{2!}) + \ldots$$

Next we use the fact that e^x is actually an infinite polynomial and cleverly arrange the terms in such a way that they may be easily summed.

$e^x(1 + v + \frac{v^2}{2!} + \ldots)$	Column #1		Column #2		Column #3		Column #4
$e^x =$	1	$+$	x	$+$	$\frac{x^2}{2!}$	$+$	$\frac{x^3}{3!} + \ldots$
$+$							
$ve^x =$			$v \cdot 1$		vx	$+$	$\frac{vx^2}{2!} + \ldots$
$+$							
$\frac{v^2}{2!}e =$					$\frac{v^2}{2!} \cdot 1$	$+$	$\frac{v^2}{2!}x + \ldots$
$+$							
$\cdot$							
$\cdot$							
$\cdot$							

$$e^x(1 + v + \frac{v^2}{2!} + \ldots) = 1 + (x + v) + \frac{1}{2}(x^2 + 2xv + v^2) + \ldots$$

$$e^x e^v = 1 + (x + v) + \frac{(x + v)^2}{2!} + \ldots$$

$$= e^{(x + v)}$$

Notice that the sum of all of the terms on the left is simply the product

$$e^x(1 + v + \frac{v^2}{2!} + \ldots)$$

or equivalently

$$(e^x)(e^v)$$

On the right we have arranged the terms so that the sum of each column will be a term in the polynomial for $e^{(x + v)}$.

PROBLEM SET 15:

1. a. Graph $y = e^x$ for the table function below.

   b. Connect the points with a smooth curve.

x	$y = e^x$
-5	.007
-4	.018
-3	.049
-2	.14
-1	.37
-.5	.60
0	1.0
.5	1.6

x	$y = e^x$
1.0	2.7
1.2	3.3
1.4	4.0
1.5	4.5
1.6	5.0
1.8	6.0
2.0	7.4
2.2	9.0
2.4	11.0
2.5	12.2
2.6	13.5
2.8	16.4
3.0	20.0

   c. Sketch in the smooth-curve graphs of $y = 2^x$ and $y = 3^x$ on the same grid. $(3^{2.5} \approx 15.6)$

2. Column #4 in Section 15-4 adds up to

$$\frac{x^3}{3!} + v\frac{x^2}{2!} + x\frac{v^2}{2!} + \frac{v^3}{3!}$$

Show that this polynomial is equivalent to

$$\frac{(x + v)^3}{3!}$$

3. Column #5 adds up to
$$\frac{x^4}{4!} + \frac{x^3v}{3!} + \frac{x^2v^2}{2!2!} + \frac{xv^3}{3!} + \frac{v^4}{4!}$$

Show that this polynomial is equivalent to

$$\frac{(x + v)^4}{4!}$$

Use the information in Problem 1 and the relation $(e^x)(e^v) = e^{x+v}$ in Problems 4 through 10. Keep in mind that the numbers in the table are only approximations. Therefore your calculations will yield numbers that are only approximately equal.

EXAMPLE:

Does $e^{(2)}e^{(-1)} = e^{(2-1)}$
$= e^1$?

SOLUTION:

$$e^2 \approx 7.4$$
$$e^{-1} \approx .37$$

Does $(7.4)(.37) = 2.718281828$?

$$2.738 \approx 2.718281828...$$

4. Does $(e^{1.2})(e^{1.8}) = e^3$?
5. Does $(e^{1.5})(e^{-.5}) = e^1$?
6. Does $(e^{2.5})(e^{-0.5}) = e^2$?
7. Does $(e^{.5})(e^{.5}) = e^1$?
8. Does $(e^{1.4})(e^{1.6}) = e^3$?
9. Does $(e^{1.8})(e^{2.2})(e^{-5}) = e^{-1}$?
10. Does $(e^{2.4})(e^{2.6})(e^{-4}) = e^1$?

11. Use the information in the table shown opposite and the equation

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

to calculate e to 5 decimal places.

n	$\frac{1}{n!}$
1	1.0
2	.5
3	.16667
4	.04167
5	.00833
6	.00139
7	.00020
8	.00002

12. a. Use the information in Problem 11 to calculate $e^{.1}$ to 7 decimal places.

b. What is $\sqrt[10]{e}$?

13. Calculate $e^{-.1}$ to several decimal places. Neglect the 7! and 8! terms.

14. Show that $\frac{d(e^x)}{dx} = e^x$ by finding the derivative of the infinite polynomial for e^x.

15. a. Find an infinite series for e^{kx} by substituting kx for x in the polynomial for e^x.

b. Find the derivative of the series in part a to show that $\frac{d(e^{kx})}{dx} = ke^{kx}$.

SECTION 16: A COMPLEX NOTION

16-1 The Meaning of $e^{i\theta}$

We are going to do something that may seem mysterious. We are going to substitute the expression $i\theta$ for x in the series for e^x. For once we will not tell you what to expect. We want you to follow the line of reasoning that we present and perhaps be surprised at the result.

Recall that

$$i = \sqrt{-1}$$

and

$$i^2 = -1$$

and so forth. You last saw "i" back in Unit III on quadratic equations. We created "i" for what probably seemed to be an artificial reason. We created it so that we could say that all quadratic equations had solutions. From the point of view of a mathematician this is a perfectly acceptable reason. In this section we will once again involve ourselves in ideas that mainly mathematicians find interesting.

Recall the series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Now we substitute $i\theta$ for x.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \ldots$$

Remember that

$$i^2 = -1, \quad i^3 = (-1)i, \quad i^4 = (i^2)^2 = (-1)^2 = 1$$

Consequently,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$$

Notice that all odd powers of θ will have an i factor with them. We can factor i out of these terms to get

$$e^{i\theta} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$
$$+ i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)$$

You should recognize both of these series. The one on the top is the series for $\cos\theta$ while the one on the bottom is the series for $\sin\theta$. These facts lead us to the rather surprising equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It is quite possible to be convinced that this equation is true and yet have absolutely no intuitive feeling for its meaning. What sense is there in a formula that relates a rather special exponential function to the trigonometric funcitons sine and cosine through the mysterious services of the elusive "i"? The first step in making sense of $e^{i\theta}$ will be to draw a picture of it.

16-2 A Graphical Interpretation of $e^{i\theta}$

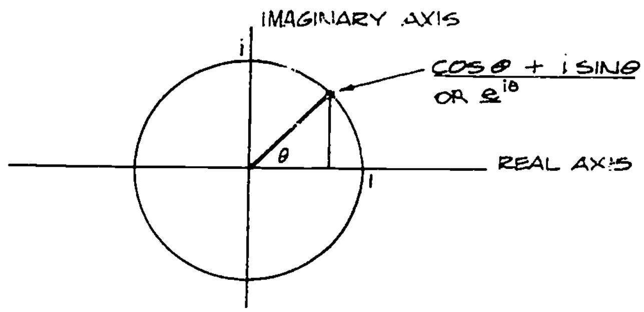
In one sense $e^{i\theta}$ is an old function. It is an old function because it is basically just an exponential funciton. Since it is an exponential function we expect it to follow the exponential properties that we have been repeating and investigating these past few sections.

In fact, we will find that e^x behaves like an exponential function even when x is a complex number. You might think that this settles the issue and that we can move on to new functions. If so, you would be wrong, because in another sense $e^{i\theta}$ is a new function. It is a new function because $e^{i\theta}$ has some new properties in addition to the exponential laws that we have developed previously.

Since $e^{i\theta}$ is a new function, one of the first things to do is to graph it. The first step in this direction is to notice that $e^{i\theta}$ is a complex number. In fact,

$$e^{i} = \cos\theta + i \sin\theta$$

The real part or horizontal coordinate is $\cos\theta$. The imaginary part or vertical coordinate is $\sin\theta$. These observations lead to the opposite graph. Notice that for any value of θ, $e^{i\theta}$ will be a point located on a circle of radius one centered at the origin. We know that the circle will have radius one because the distance from the origin will be given by the Pythagorean Theorem. Specifically,



$$\text{distance} = \sqrt{\cos^2\theta + \sin^2\theta}$$

and we know that

$$\cos^2\theta + \sin^2\theta = 1$$

for all values of θ: therefore

$$\text{distance} = \sqrt{1}$$
$$= 1$$

16-3 Some Calculations with $e^{i\theta}$

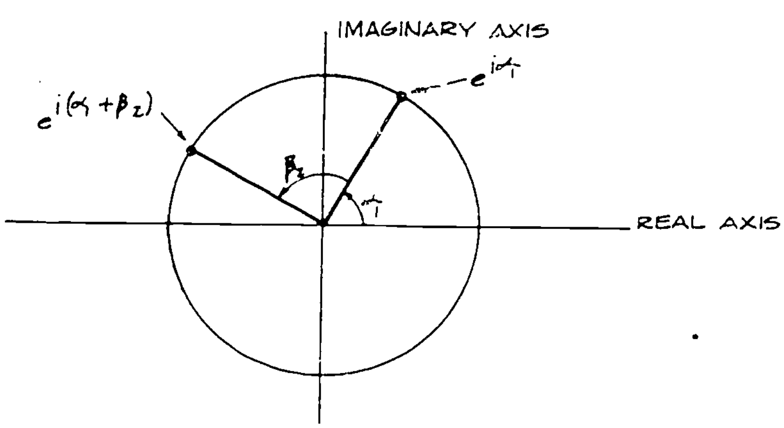
Now we will look into the matter of multiplying two complex numbers written in the form $e^{i\theta}$. For example,

$$(e^{i\alpha})(e^{i\beta}) = ?$$

Since $e^{i\theta}$ is an exponential function we expect it to follow the $(e^x)(e^y) = e^{x+y}$ rule; therefore,

$$(e^{i\alpha})(e^{i\beta}) = e^{i\alpha + i\beta}$$
$$= e^{i(\alpha + \beta)}$$

We can graphically represent this operation as follows.



16-4 A Numerical Example

Remember the formulas

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

required that x be stated in radians.

EXAMPLE:

What is the complex number represented by $e^{i\frac{\pi}{4}}$?

SOLUTION:

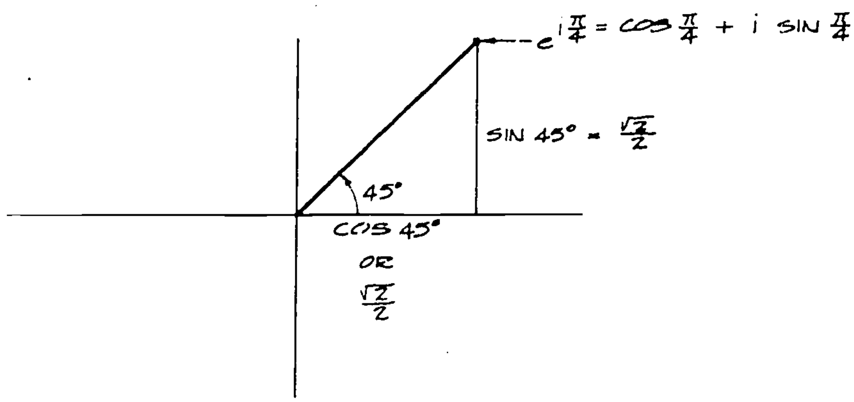
We apply the relation $e^{i\theta} = \cos\theta + i \sin\theta$ and substitute $\theta = \frac{\pi}{4}$ to obtain

$$e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i \sin \frac{\pi}{4}$$

Recall that $\frac{\pi}{4}$ radians is equivalent to 45°; therefore

$$e^{i\frac{\pi}{4}} = \cos 45° + i \sin 45°$$

Graphically represented, this is



In summary then,

$$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

PROBLEM SET 16:

1. Show that $e^{i\theta} = \cos\theta + i \sin\theta$ by substituting $i\theta$ for x in the power series for e^x.

Convert the following angles to radian measure. Use the conversion factor

$$1 = \frac{\pi \text{ radians}}{180°}$$

2. 90°
3. 180°
4. 45°
5. 360°
6. 60°
7. 135°

Refresh your memory about trigonometric functions by finding the values of the following functions.

8. $\cos 0 = ?$ 10. $\cos \frac{\pi}{2} = ?$ 12. $\sin \pi = ?$ 14. $\cos \frac{\pi}{4} = ?$ 16. $\sin \frac{\pi}{6} = ?$

9. $\sin 0 = ?$ 11. $\sin \frac{\pi}{2} = ?$ 13. $\cos \pi = ?$ 15. $\sin \frac{\pi}{4} = ?$ 17. $\sin \frac{\pi}{3} = ?$

Graph the complex numbers represented by the following imaginary powers of e.

18. $e^{i\frac{\pi}{4}}$ 19. $e^{i\frac{\pi}{2}}$ 20. $e^{i\frac{\pi}{3}}$ 21. $e^{i\frac{\pi}{6}}$ 22. $e^{i\frac{5\pi}{6}}$ 23. $e^{i\frac{3\pi}{2}}$

Represent the following multiplications graphically.

24. $(e^{i\frac{\pi}{2}})(e^{i\pi})$ 25. $(e^{i\frac{\pi}{3}})(e^{i\frac{3\pi}{4}})$

Write the complex numbers represented by the imaginary powers of e given below.

26. $e^{i\frac{\pi}{4}} =$ 27. $e^{i\frac{\pi}{2}} =$ 28. $e^{i\pi} =$ 29. $e^{i\frac{3\pi}{4}} =$

*30. Raise $e^{i\frac{\pi}{2}}$ to the ith power to get an expression for i^i. Describe anything notable about the resulting expression.

SECTION 17: THE POLAR FORM OF A COMPLEX NUMBER

17-1 Finding the Polar Form

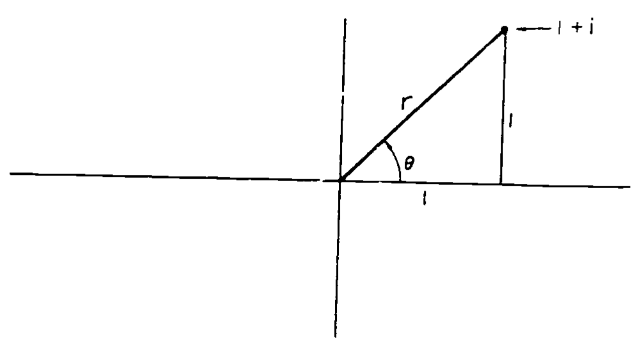
In the preceding section we saw that the graph of $e^{i\theta}$ was a circle of radius one in the complex plane. Here we will see that we can represent any point in the complex plane by an expression of the form $re^{i\theta}$ where r is the distance of the point from the origin.

EXAMPLE:

Write the number $1 + i$ in the form $re^{i\theta}$.

SOLUTION:

A graph of $1 + i$ will tell us much.



The Pythagoream Theorem will tell us the distance from the origin.

$$r = \sqrt{1^2 + 1^2}$$

$$r = \sqrt{2}$$

Our earlier experience with trigonometry tells us that the angle θ is 45°, or $\frac{\pi}{4}$ radians. Putting it all together, we get

$$1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

It should be clear that a similar procedure could be used to convert any complex number in the form a + bi into the polar form $re^{i\theta}$.

17-2 Multiplication and Division of Complex Numbers Revisited

When two complex numbers are stated in polar form, the operations of multiplication and division are greatly simplified. The following numerical examples illustrate the new procedures.

MULTIPLICATION EXAMPLE:

$$(3e^{i})(5e^{2i}) = ?$$

SOLUTION:

We use the commutative principle to rearrange the factors.

$$(3e^{i})(5e^{2i}) = 15(e^{i})(e^{2i})$$

Now look at the product of the two exponential terms. We can apply the property

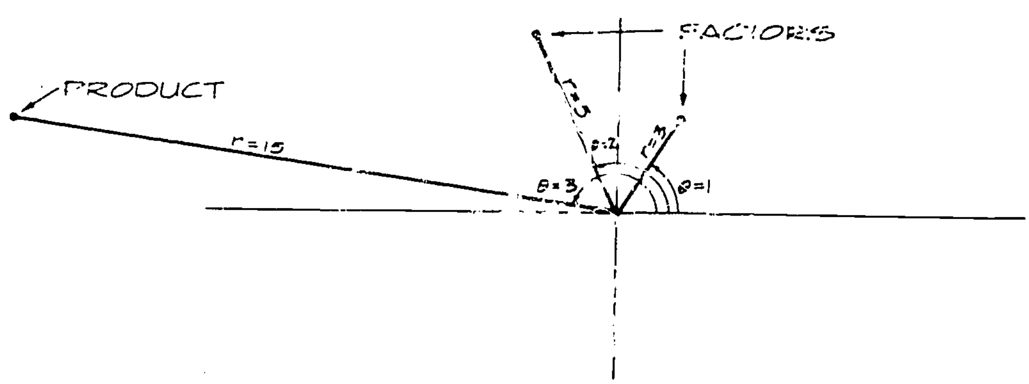
$$(e^{x})(e^{y}) = e^{x+y}$$

to obtain

$$(3e^{i})(5e^{2i}) = 15e^{(i+2i)}$$
$$= 15e^{3i}$$

In other words, the product of $(3e^{i})(5e^{2i})$ will be 15 units away from the origin and the angle θ will be 3 radians.

A graph of this problem will show what is happening from another point of view.



Notice that the length of the product (15) is the product of the lengths of the factors (3·5), while the angle of the product (3 radians) is the sum of the angles of the factors (1 radian plus 2 radians). This pattern is true in general for multiplication.

DIVISION EXAMPLE:

$$\frac{12e^{i\pi}}{3e^{i\frac{\pi}{3}}} = ?$$

First of all we can cancel a factor of three from both the denominator and numerator.

$$\frac{\cancelto{4}{12}e^{i\pi}}{\cancel{3}e^{i\frac{\pi}{3}}} = \frac{4e^{i\pi}}{e^{i\frac{\pi}{3}}}$$

Next we apply the $e^x = \frac{1}{e^{-x}}$ to get

$$= 4(e^{i\pi})(e^{-i\frac{\pi}{3}})$$

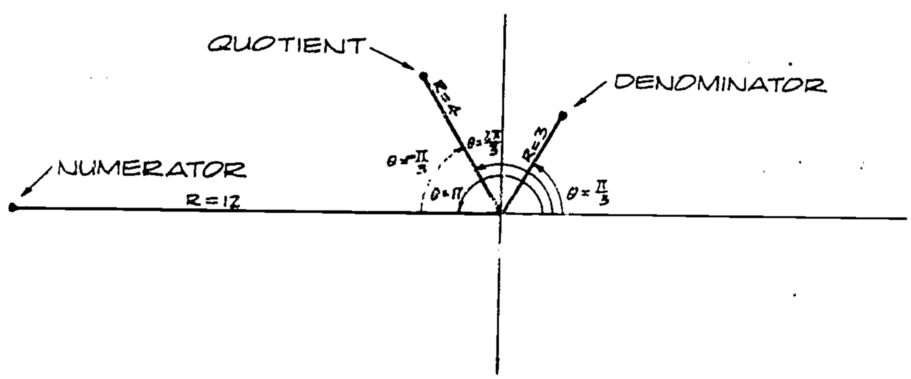
Now we use the rule $(e^x)(e^y) = e^{x+y}$ to go on to

$$= 4e^{(i\pi - i\frac{\pi}{3})}$$

$$= 4e^{i(\pi - \frac{\pi}{3})}$$

$$= 4e^{i\frac{2}{3}\pi}$$

Again we will refer to a graph of this procedure to show the general pattern in division.



We can see that the length of the quotient (4) is the quotient of the numerator (12) divided by the denominator (3), while the angle of the quotient ($\frac{2\pi}{3}$ radians) is the angle of the numerator (π radians) minus the angle of the denominator ($\frac{\pi}{3}$ radians). This is the general pattern for division.

Compare these procedures for multiplying and dividing complex numbers to the ones we first used (Section 21, Unit III). When we were developing arithmetic procedures for complex numbers back then, we were able to draw a parallel between vectors and complex numbers for the operations of addition and subtraction. However, the pattern for complex multiplication was not obvious in the procedure that we used. For example, consider the product

$$(1+i)(2+2i) = 2 + 2i + 2i + 2i^2$$

$$= 4i$$

The pattern is unintuitive when it's presented in this form. It is not obvious that the lengths of the two complex numbers are being multiplied and the angles being added.

PROBLEM SET 17:

Convert the following complex numbers to polar form.

1. $-1 + i$ 2. $-1 - i$ 3. $1 + i\sqrt{3}$ 4. $-1 + i\sqrt{3}$

5. i 6. -1 7. $-\sqrt{3} + i$

Use a protractor and a ruler to sketch the following complex multiplications and divisions. Show the value of the product on the sketch.

8. $(4e^{i\frac{\pi}{2}})(3e^{i\frac{\pi}{4}})$ 9. $(6e^{i\frac{\pi}{3}})(3e^{i\frac{5}{6}\pi})$ 10. $\dfrac{16e^{i\frac{7\pi}{6}}}{8e^{i\frac{5\pi}{6}}}$ 11. $\dfrac{20e^{i\frac{\pi}{6}}}{5e^{i\frac{\pi}{3}}}$

Perform the following indicated operations.

12. $(\pi e^{i\frac{\pi}{4}})(\frac{1}{\pi}e^{i\frac{7\pi}{4}}) =$ 15. $(\frac{2}{3}e^{.7i})(\frac{3}{8}e^{-.4i}) =$ 18. $\dfrac{49e^{16i}}{14e^{-7i}} =$

13. $(6e^{80i})(.5e^{75i}) =$ 16. $(.8e^{-.4i})(.12e^{.6i}) =$ 19. $\dfrac{39e^{21i}}{13e^{47i}}$

14. $(14e^{.05i})(10e^{.78i}) =$ 17. $\dfrac{27e^{6i}}{9e^{7i}} =$ 20. $\dfrac{6e^{-19i}}{18e^{-21i}}$

REVIEW PROBLEM SET 18:

Convert the following equations (solve for y)

1. $\log y = 3 \log x + 2$ 3. $\log y = -1.8 \log t + 1.94$

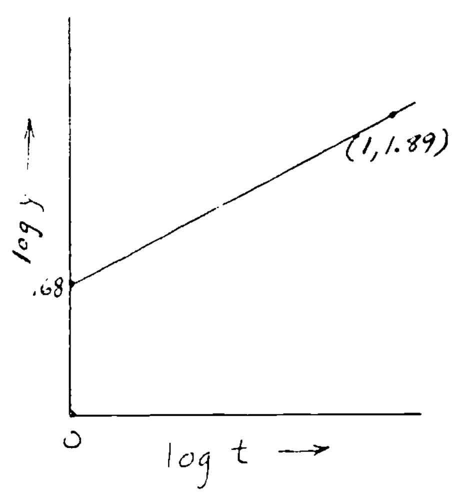
2. $\log y = .52 \log t + .58$ 4. $\log y = -3.5 \log x - .02$

In each of Problems 5 through 7 find:

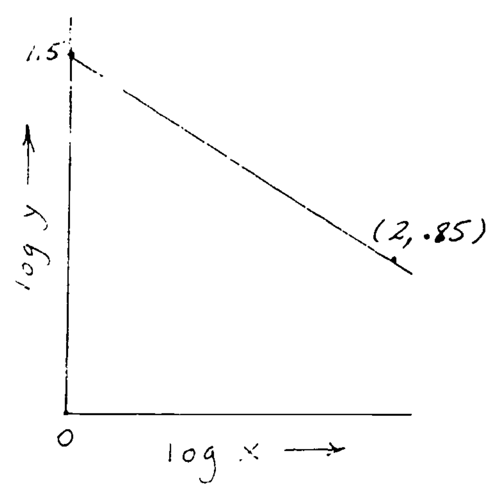
a. the equation of the line.

b. the converted equation (solve for y as in Problems 1 through 4).

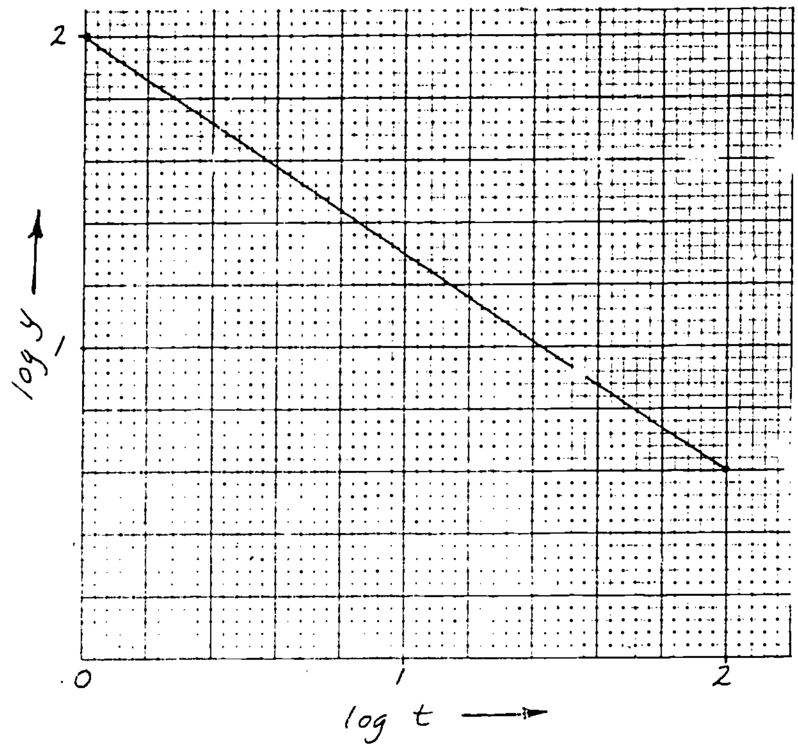
5.



6.



7.



8. The relationship between the mass M of an animal in kg and the heart rate R in beats per minute is

$$\log R = -.25 \log M + 2.3$$

What heart rate would you predict for Hercules, a dog of [illegible]2 kg? Round to the nearest whole heartbeat.

Find the sum of each of the following infinite series.

9. 8.1 + .81 + .081 + .0081 + ...

10. .63 + .063 + .0063 + ...

11. .27 + .027 + .0027 + ...

12. .333 + .0333 + .00333 + .000333 + ...

13. 5.22 + .522 + .0522 + .00522 + ...

Recall the infinite series for the sine function

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Here, x is given in <u>radians</u>.

14. Use the first two terms of the series to get an approximate value for sin x, for each of the following values of x.

a. x = .1 b. x = .01 c. x = .001

15. How do the approximate values of sin x in Problem 14 compare with the true values? Use the table on the following page.

Table of sines for small values of x

x(radians)	Sin x
0.000	0.00000 00000
0.001	0.00099 99998
0.002	0.00199 99987
0.003	0.00299 99955
0.004	0.00399 99893
0.005	0.00499 99792
0.006	0.00599 99640
0.007	0.00699 99428
0.008	0.00799 99147
0.009	0.00899 98785
0.010	0.00999 98333
0.020	0.01999 86667
0.030	0.02999 55002
0.040	0.03998 93342
0.050	0.04997 91693
0.060	0.05996 40065
0.070	0.06994 28473
0.080	0.07991 46940
0.090	0.08987 85492
0.100	0.09983 34166

Find the complex numbers represented by the following imaginary powers of e.

16. $e^{i\frac{5}{6}\pi}$ 18. $e^{i\frac{3}{2}\pi}$ 20. $e^{i\frac{5}{3}\pi}$ 22. $2e^{i\frac{\pi}{3}}$

17. $e^{i\frac{4}{3}\pi}$ 19. $e^{i\frac{7\pi}{4}}$ 21. $e^{i\frac{11}{6}\pi}$ 23. $\sqrt{2}e^{i\frac{5}{4}\pi}$

Convert the following complex numbers to polar form.

24. $1 - i$ 25. $\sqrt{3} - i$ 26. $1 - i\sqrt{3}$ 27. $-\sqrt{3} - i$

Perform the indicated operations.

28. $(2e^{i\pi})(3e^{i\frac{\pi}{3}})$

29. $(7e^{i\frac{\pi}{4}})(3e^{i\frac{\pi}{6}})$

30. $(5e^{1.6i})(2e^{1.8i})$

31. $\dfrac{8e^{i\frac{\pi}{2}}}{2e^{i\frac{\pi}{3}}}$

32. $\dfrac{12e^{.7i}}{[illegible]e^{.3i}}$

LOG TABLE

y	log y	y	log y	y	log y
1.00	.000	4.00	.602	7.00	.845
1.05	.021	4.05	.607	7.05	.848
1.10	.041	4.10	.613	7.10	.851
1.15	.061	4.15	.618	7.15	.854
1.20	.079	4.20	.623	7.20	.857
1.25	.097	4.25	.628	7.25	.860
1.30	.114	4.30	.633	7.30	.863
1.35	.130	4.35	.638	7.35	.866
1.40	.146	4.40	.643	7.40	.869
1.45	.161	4.45	.648	7.45	.872
1.50	.176	4.50	.653	7.50	.875
1.55	.190	4.55	.658	7.55	.878
1.60	.204	4.60	.663	7.60	.881
1.65	.217	4.65	.667	7.65	.884
1.70	.230	4.70	.672	7.70	.886
1.75	.243	4.75	.677	7.75	.889
1.80	.255	4.80	.681	7.80	.892
1.85	.267	4.85	.686	7.85	.895
1.90	.279	4.90	.690	7.90	.898
1.95	.290	4.95	.695	7.95	.900
2.00	.301	5.00	.699	8.00	.903
2.05	.312	5.05	.703	8.05	.906
2.10	.322	5.10	.708	8.10	.908
2.15	.[illegible]32	5.15	.712	8.15	.911
2.20	.342	5.20	.716	8.20	.914
2.25	.352	5.25	.720	8.25	.916
2.30	.362	5.30	.724	8.30	.919
2.35	.371	5.35	.728	8.35	.922
2.40	.380	5.40	.732	8.40	.924
2.45	.3[illegible]9	5.45	.736	8.45	.927
2.50	.398	5.50	.740	8.50	.929
2.55	.407	5.55	.744	8.55	.932
2.60	.415	5.60	.748	8.60	.934
2.65	.423	5.65	.752	8.65	.937
2.70	.[illegible]31	5.70	.756	8.70	.940
2.75	.439	5.75	.760	8.75	.942
2.80	.447	5.80	.763	8.80	.944
2.85	.455	5.85	.767	8.85	.947
2.90	.462	5.90	.771	8.90	.949
2.95	.470	5.95	.775	8.95	.952
3.00	.477	6.00	.778	9.00	.954
3.05	.484	6.05	.782	9.05	.957
3.10	.491	6.10	.785	9.10	.959
3.15	.498	6.15	.789	9.15	.961
3.20	.505	6.20	.792	9.20	.964
3.25	.512	6.25	.796	9.25	.966
3.30	.519	6.30	.799	9.30	.968
3.35	.525	6.35	.803	9.35	.971
3.40	.531	6.40	.806	9.40	[illegible]
3.45	.538	6.45	.810	9.45	[illegible]
3.50	.544	6.50	.813	9.50	[illegible]
3.55	.550	6.55	.816	9.55	.980
3.60	.556	6.60	.820	9.60	.982
3.65	.562	6.65	.823	9.65	.985
3.70	.568	6.70	.826	9.70	.987
3.75	.574	6.75	.829	9.75	.989
3.80	.580	6.80	.833	9.80	.991
3.85	.585	6.85	.836	9.85	.993
3.90	.591	6.90	.839	9.90	.996
3.95	.597	6.95	.842	9.95	.998
				10.00	1.000